EECS2101 Fundamentals of Data Structures

Lecture Notes

Winter 2025 (Section Z)

Jackie Wang

Lecture 1 - January 7

Syllabus
Introduction to the Course

Solving Problems via Data Structures

Course Learning Outcomes (CLOs)

CLO1 Instantiate a range of standard abstract data types (ADT) as data structures.

CLO2 Implement these data structures and associated operations and check that they satisfy the properties of the ADT.

CLO3 Apply best practice software engineering principles in the design of new data structures.

CLO4 Demonstrate the ability to reason about data structures using contracts, assertions, and invariants.

CLO5 Analyse the asymptotic run times of standard operations for a broad range of common data structures.

CLO6 Select the most appropriate data structures for novel applications.

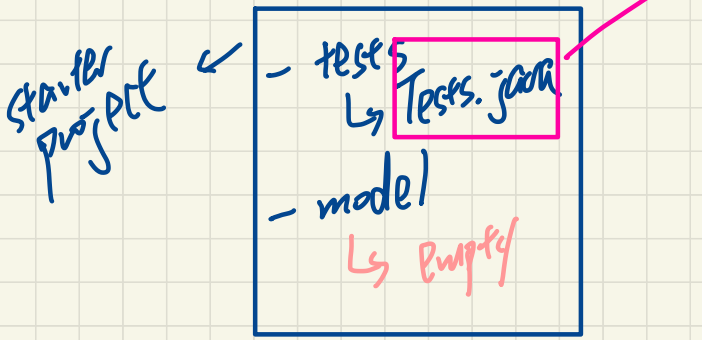
Written Test

- ~ Section specific
- ~ eClass (in-person)
- ~ multiple choice questions (one or more correct answers)

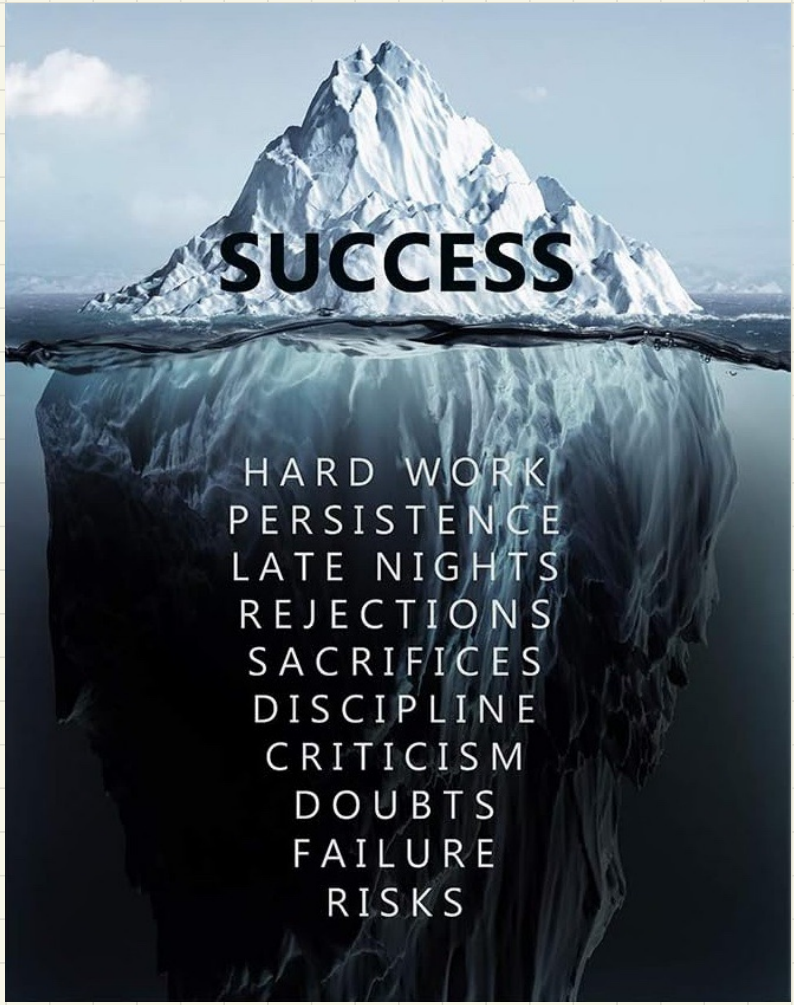
Exam

- in-person
- 3 hours
- mostly written questions.

Programming Tests



1. example usages of methods
2. meant to be incomplete
   - ↳ You're expected:
     - (1) not to make your code work only for starter tests.
     - (2) write additional tests



General Tips about Success

Source: https://a.co/d/aQ13fR1

Lecture 2 - January 9

Introduction to the Course
Recursion: Part 1

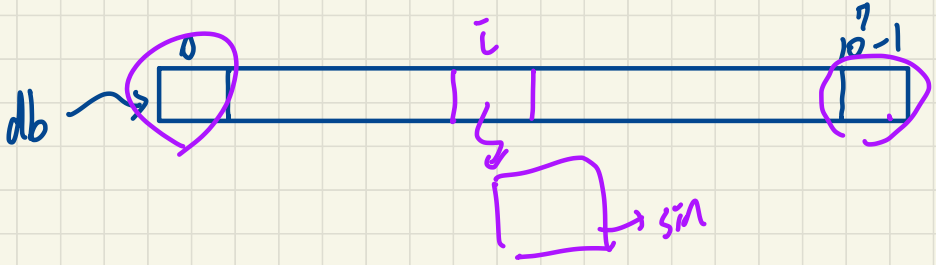
Solving Problems via Data Structures
References to Recursion Basics
More Advanced Recursion: splitArray

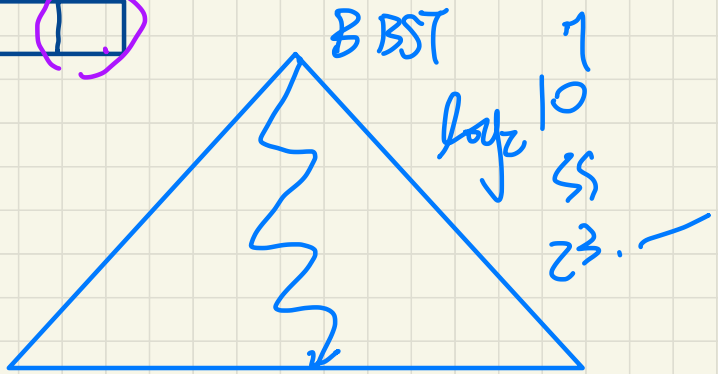
Announcements/Reminders

- Assignment 1 to be released
- Office Hours: 3pm to 4pm, Mon/Tue/Wed/Thu
- Trial attendance check via iClicker today!

A Searching Problem

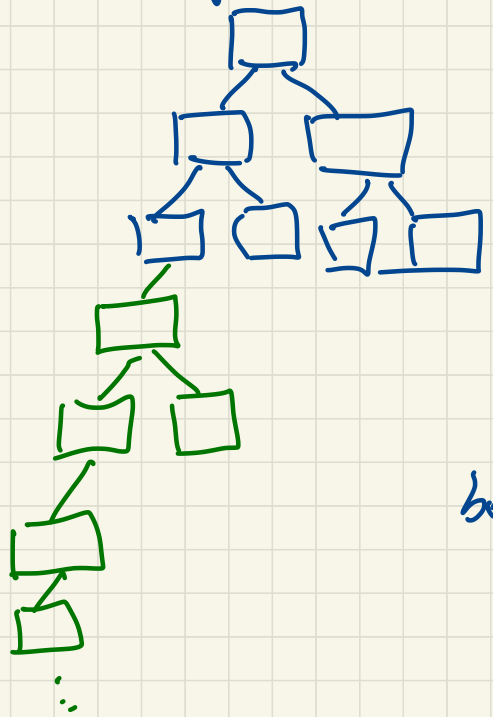
```
ResidentRecord find(int sin) {
  for(int i = 0; i < database.length; i ++) {
    if(database[i].sin == sin) {
      return database[i];
    }
  }
}
```

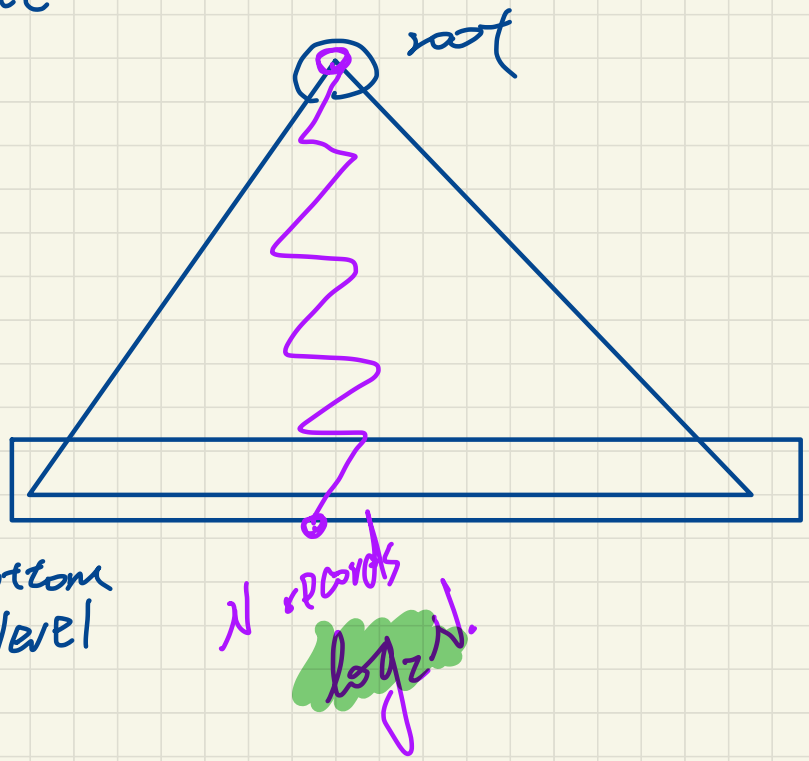




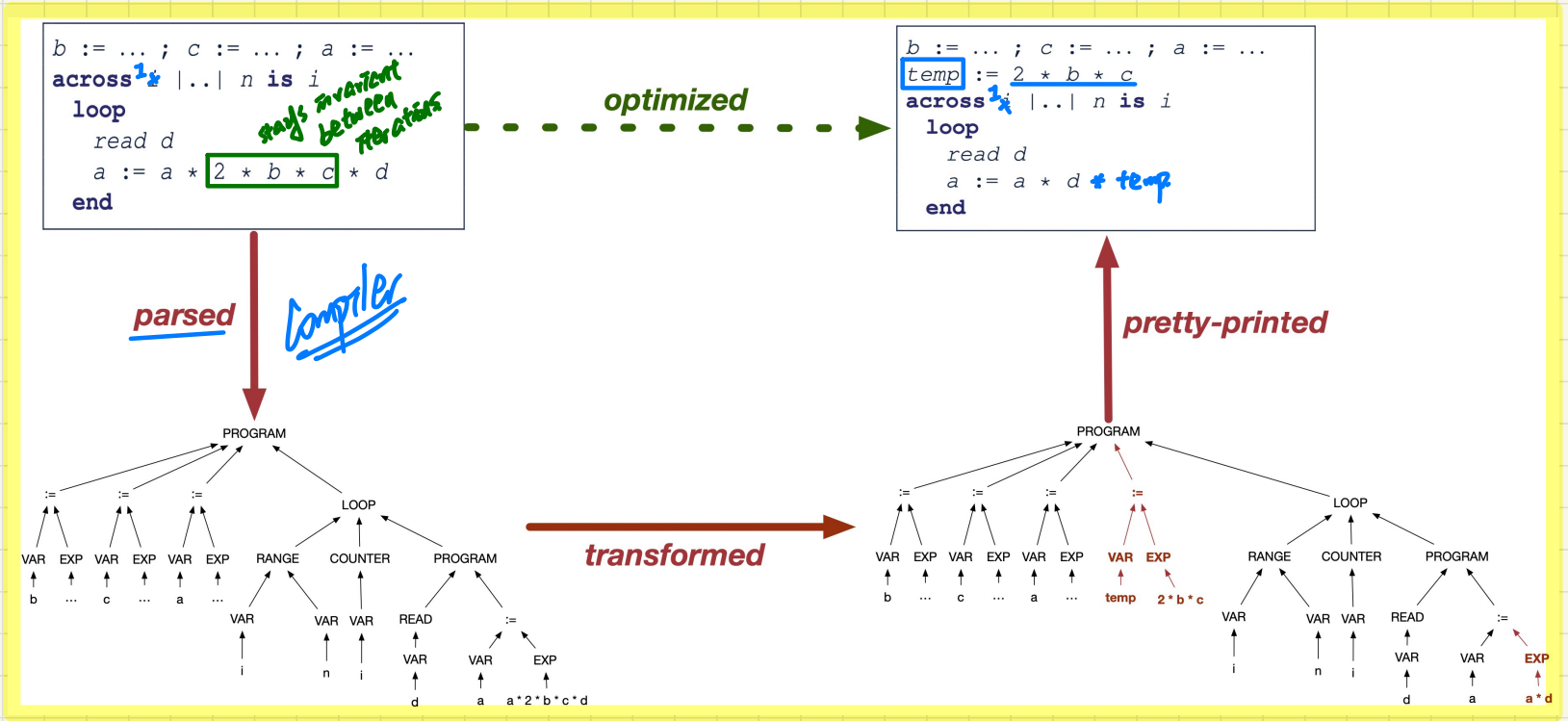
Solution for Efficient Searching

Balanced Binary Search Tree

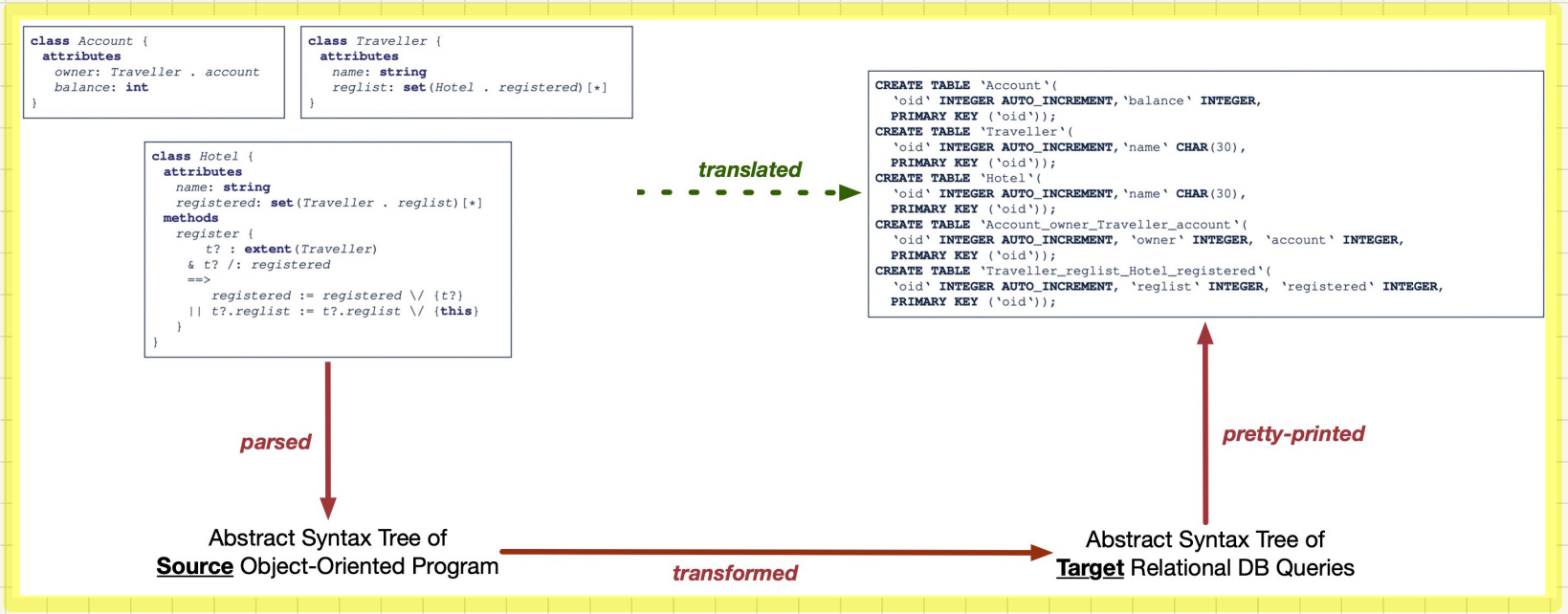




Program Optimization Problem



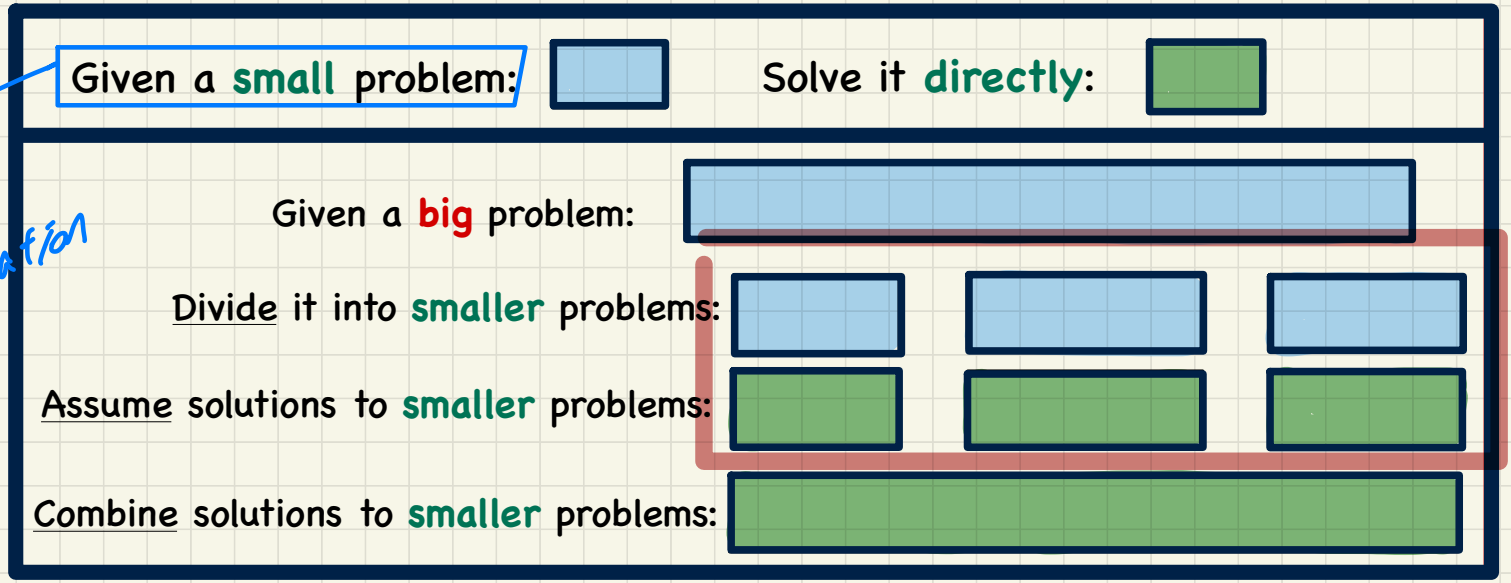
Program Translation Problem



Solving a Problem Recursively

P 1 ... ∞ Mathematical Induction (Strong)

P(1) P(2) Assume: P(k) P(k+1) show: P(n)



Base cases ensure termination

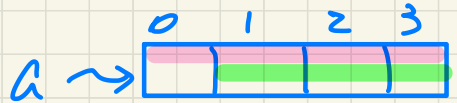
```
m(i) {
  if(i == ...) { /* base case: do something directly */ }
  else {
    m(j);/* recursive call with strictly smaller value */
  }
}
```

Recursion on an Array: Passing Same Array Reference

```
void m(int[] a, int from, int to) {
  if(from > to) { /* base case */ }
  else if(from == to) { /* base case */ }
  else { m(a, from + 1, to) } }
```

specify range of elements to consider in 'a'

call by value

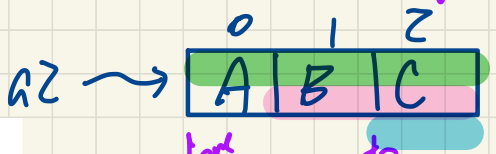


from 0 to 3

from 1 to 3

1 0 empty range.

Say a1 = {}, consider m(a1, 0, a1.length - 1)



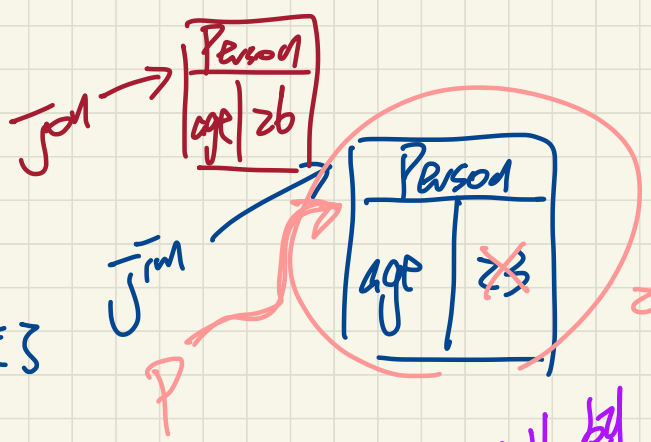
Say a2 = {A, B, C}, consider m(a2, 0, a2.length - 1)

m(a2, 0, 2)

m(a2, 1, 2)

m(a2, 2, 2) → base case

```
class Person {
    int age;
    Person(int age) { ... }
    void getOlder(int i) { age += i; }
}
```




```
Person jim = new Person(23);
Person jon = new Person(jim.age);
```

```
class Util {
    void changeAge(Person p) {
        p.getOlder(3);
    }
}
```

parameter

call by value:

```
p = jim;
```

```
Util u = new Util();
u.changeAge(jim);
assertEquals(26, jim.age);
```

argument

Problem on Recursion

https://codingbat.com/prob/p185204

Given an array of ints, is it possible to divide the ints into two groups, so that the sums of the two groups are the same. Every int must be in one group or the other. Write a recursive helper method that takes whatever arguments you like, and make the initial call to your recursive helper from splitArray(). (No loops needed.)

splitArray([2, 5, 3]) → t

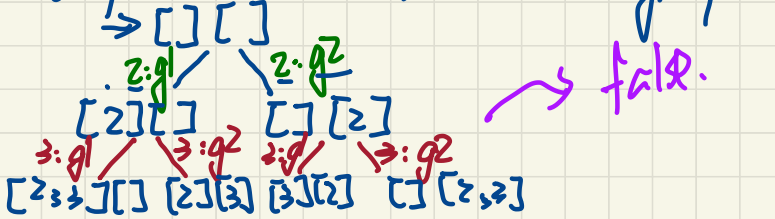
splitArray([2, 2]) → true

splitArray([2, 3]) → false

splitArray([5, 2, 3]) → true

Intuition

How many ways to put n elements into m groups?



Lecture 3 - January 14

Recursion: Part 1

splitArray: Implementation and Tracing

Announcements/Reminders

- Assignment 1 release
- Office Hours: 3pm to 4pm, Mon/Tue/Wed/Thu
- Contact Information of TAs on common eClass site

Problem on Recursion

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Given an array of ints, is it possible to divide the ints into two groups, so that the sums of the two groups are the same. Every int must be in one group or the other. Write a recursive helper method that takes whatever arguments you like, and make the initial call to your recursive helper from splitArray(). (No loops needed.)

splitArray([2, 5, 3]) → t

splitArray([2, 2]) → true

splitArray([2, 3]) → false

splitArray([5, 2, 3]) → true

Insight: The recursive & base cases are defined s.t. the tree of recursive calls represents an exhaustive search of all possible input scenarios.

worst case

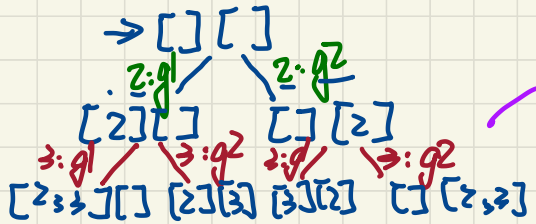
n elements

$m \times m \times \cdots \times m$

m^n

Intuition

How many ways to put n elements into m groups?



→ false.

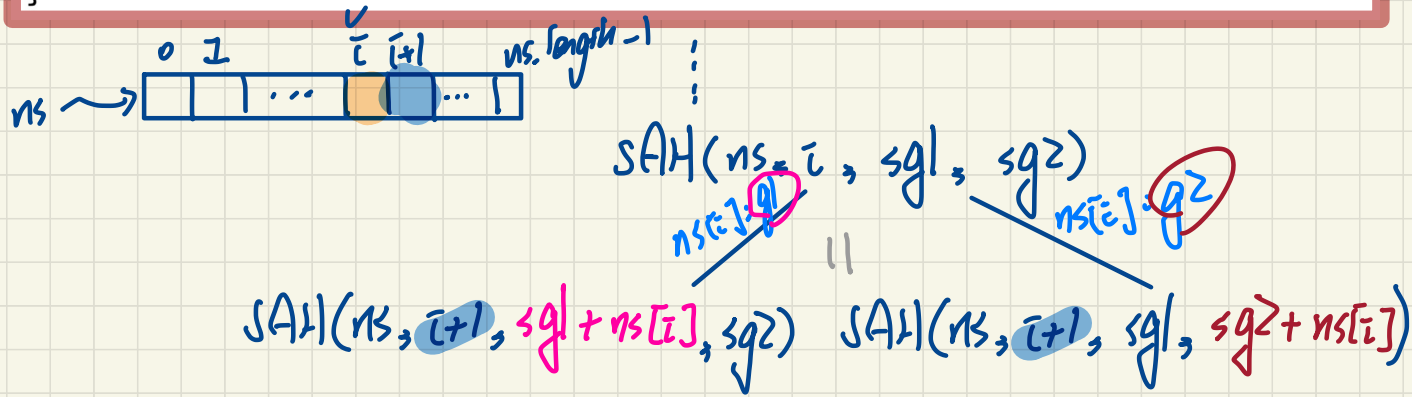
splitArray: Java Implementation

```java
public boolean splitArray(int[] ns) {
    return splitArrayHelper(ns, 0, 0, 0);
}
```

empty groups

starting index to inspect : left to right

```java
private boolean splitArrayHelper(int[] ns, int i, int sumOfGroup1, int sumOfGroup2) {
    if(i == ns.length) {
        return sumOfGroup1 == sumOfGroup2;
    }
    else {
        return
            splitArrayHelper(ns, i + 1, sumOfGroup1 + ns[i], sumOfGroup2)
            ||
            splitArrayHelper(ns, i + 1, sumOfGroup1, sumOfGroup2 + ns[i]);
    }
}
```



Math

Commutativity: $p \wedge q \equiv q \wedge p$

$p \vee q \equiv q \vee p$

(2.2) p eval. to F → also eval. q

p eval. to T → skip q → overall T

Java

Evaluation order matters: (2.1) $p \;\&\&\; q \neq q \;\&\&\; p$

① Evaluation: left to right

(2.1) p eval. to T → eval q

p eval to F → skip q → overall F

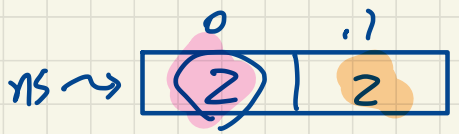
(2.2) ✓ $p \;||\; q \neq q \;||\; p$

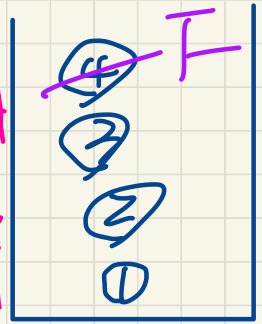
splitArray: Tracing (1)

```
public boolean splitArray(int[] ns) {
    return splitArrayHelper(ns, 0, 0, 0);
}
```

```
@Test
public void testSplitArray_01() {
    RecursiveMethods rm = new RecursiveMethods();
    int[] input = {2, 2};
    assertEquals(true, rm.splitArray(input));
}
```

```
private boolean splitArrayHelper(int[] ns, int i, int sumOfGroup1, int sumOfGroup2) {
    if(i == ns.length) {
        return sumOfGroup1 == sumOfGroup2;
    }
    else {
        return
            splitArrayHelper(ns, i + 1, sumOfGroup1 + ns[i], sumOfGroup2)
            ||
            splitArrayHelper(ns, i + 1, sumOfGroup1, sumOfGroup2 + ns[i]);
    }
}
```





splitArray: Tracing (2)

```java
public boolean splitArray(int[] ns) {
    return splitArrayHelper(ns, 0, 0, 0);
}
```

```java
private boolean splitArrayHelper(int[] ns, int i, int sumOfGroup1, int sumOfGroup2) {
    if(i == ns.length) {
        return sumOfGroup1 == sumOfGroup2;
    }
    else {
        return
            splitArrayHelper(ns, i + 1, sumOfGroup1 + ns[i], sumOfGroup2)
            ||
            splitArrayHelper(ns, i + 1, sumOfGroup1, sumOfGroup2 + ns[i]);
    }
}
```

```java
@Test
public void testSplitArray_04() {
    RecursiveMethods rm = new RecursiveMethods();
    int[] input = {5, 2, 2};
    assertEquals(false, rm.splitArray(input));
}
```

Exercises

(1) Trace on paper

(2) Trace on Eclipse.

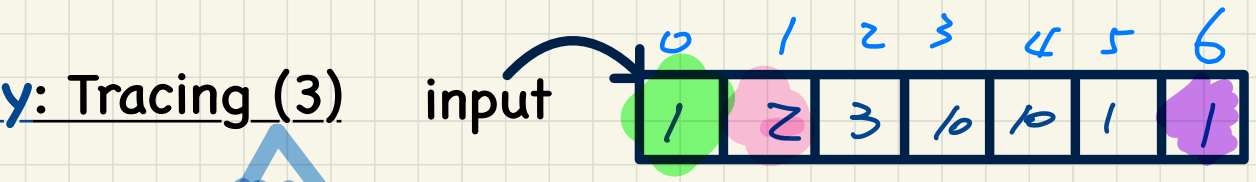
splitArray: Tracing (3)

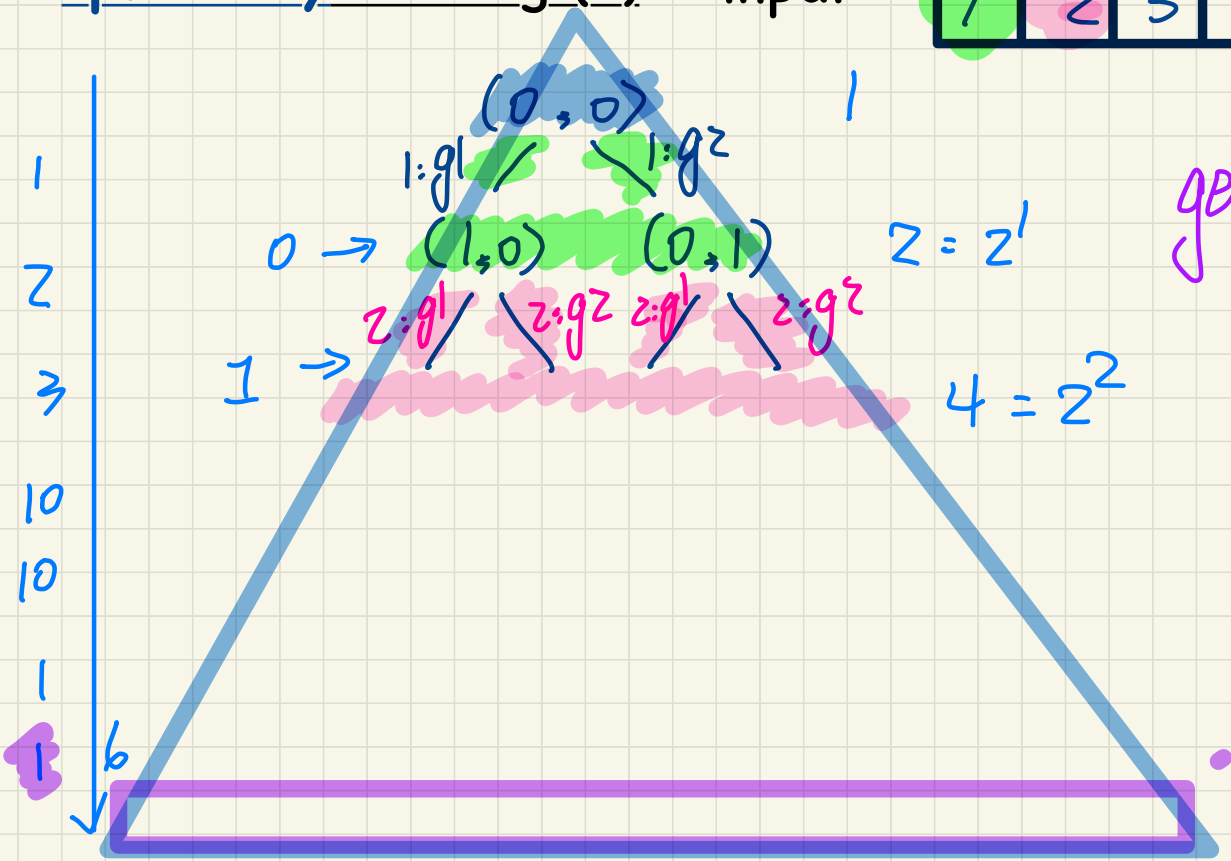
```
@Test
public void testSplitArray_13() {
    RecursiveMethods rm = new RecursiveMethods();
    int[] input = {1, 2, 3, 10, 10, 1, 1};
    assertEquals(true, rm.splitArray(input));
}
```

```
public boolean splitArray(int[] ns) {
    return splitArrayHelper(ns, 0, 0, 0);
}
```

```
private boolean splitArrayHelper(int[] ns, int i, int sumOfGroup1, int sumOfGroup2) {
    if(i == ns.length) {
        return sumOfGroup1 == sumOfGroup2;
    }
    else {
        boolean possibility1 = splitArrayHelper(ns, i + 1, sumOfGroup1 + ns[i], sumOfGroup2);
        boolean possibility2 = splitArrayHelper(ns, i + 1, sumOfGroup1, sumOfGroup2 + ns[i]);
        return possibility1 || possibility2;
    }
}
```

splitArray: Tracing (3)





1

$2 = 2^1$

geometrical seq.

$4 = 2^2$

$2^7 = 128$

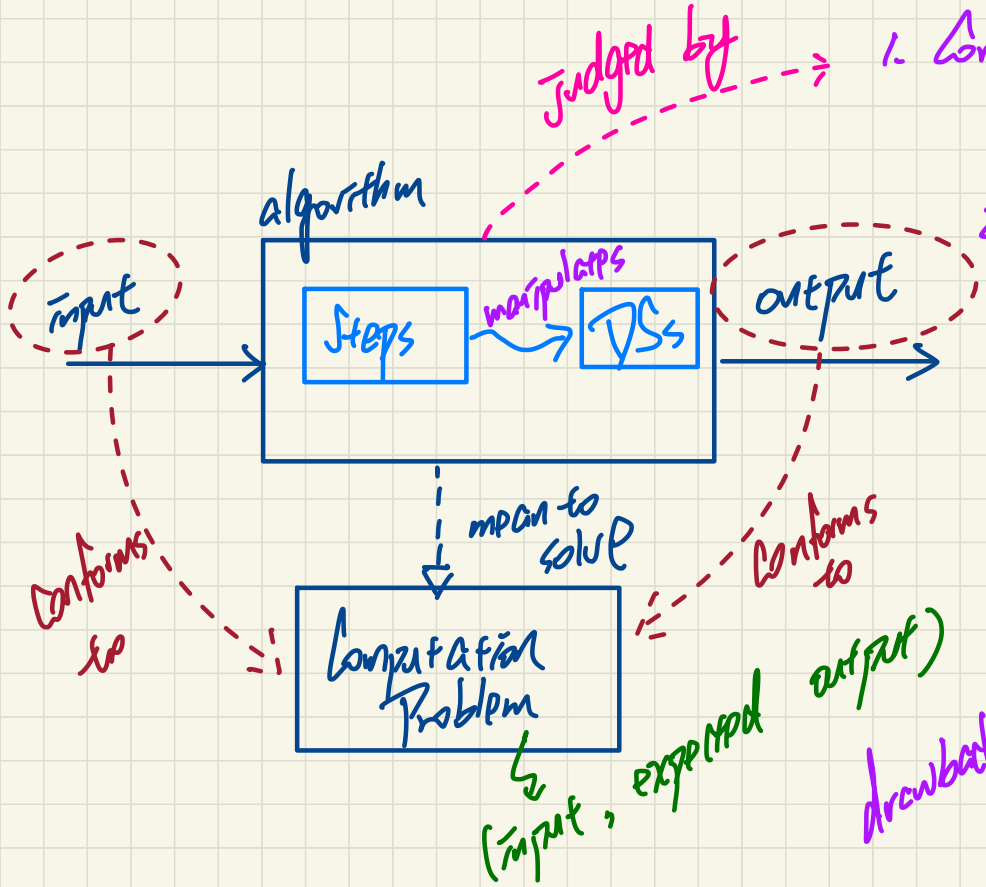
Lecture 4 - January 16

Asymptotic Analysis of Algorithms

Limitations of Experiments
Primitive Operations (POs)
Counting POs: findMax

Announcements/Reminders

- **Assignment 1** released
- Office Hours: 3pm to 4pm, Mon/Tue/Wed/Thu
- Contact Information of TAs on common eClass site
- ***splitArrayHarder***: an **extended** version coming soon



1. Correctness → Testing (dynamic)
   → Formal (static) (3342, 4315)
2. Efficiency
   → Time → Experiments
   → Math Funct (big-O)
   Space → may impact RT

drawbacks
1. full implementation (working)
2. exec. environment
3. input representativeness

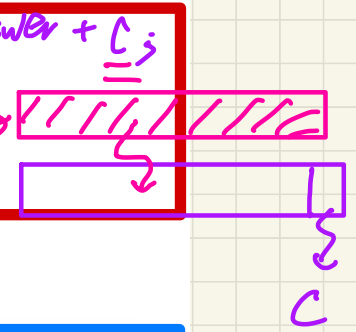
Example Experiment

Computational Problem:

- **Input**: A character *c* and an integer *n*
- **Output**: A string consisting of *n* repetitions of character *c*
  e.g., Given input `'*'` and `15`, output `***************`.

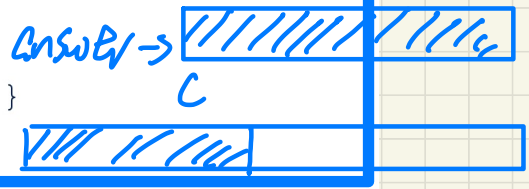
Algorithm 1 using *String* Concatenations:

```
public static String repeat1(char c, int n) {
  String answer = "";
  for (int i = 0; i < n; i ++) { answer += c; }
  return answer; }
```



Algorithm 2 using *StringBuilder* `append`'s:

```
public static String repeat2(char c, int n) {
  StringBuilder sb = new StringBuilder();
  for (int i = 0; i < n; i ++) { sb.append(c); }
  return sb.toString(); }
```



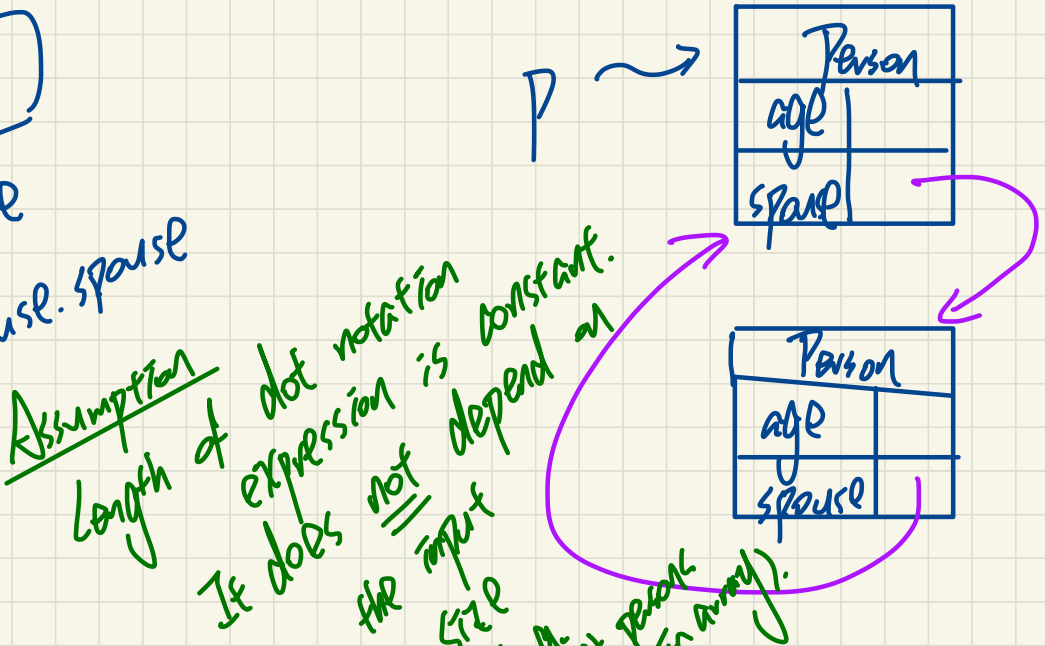
Accessing an object's attribute

```
Person p = new Person(...);
```

p.age

p.spouse

p.spouse.spouse



Assumption

Length of dot notation expression is constant.

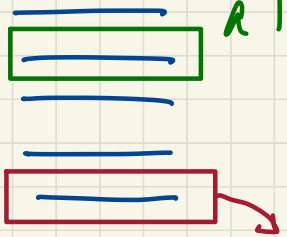
If does not depend on the input size (e.g. # person in array).

Method Call : may or may not be a PO

obj.m(...)

Case 1 : m considered primitive

m() {



a PO

}

can be (1) a method call (that's considered a PO) (2) a loop

Case 2 m not primitive

m() {

for(int i=0; i<n; i++){ ...

}

}

size of input array

findMax(a, a.length)

i<n && i>2 → 2~n.

Example 1: Counting Number of Primitive Operations

```
1  int findMax (int[] a, int n) {
2    currentMax = a[0];
3    for (int i = 1; i < n; ) {
4      if (a[i] > currentMax) {
5        currentMax = a[i]; }
6      i ++ }
7    return currentMax; }
```

i	incl. i<n
1	true
2	true
⋮	
n-1	true
n	false

Q. # of times `i < n` in **Line 3** is executed?

n times (n-1 times i<n → true ; 1 time i<n → false)

Q. # of times **loop body (Lines 4 to 6)** is executed?

n-1 times (when i<n evaluates to T).

Lecture 5 - January 21

Asymptotic Analysis of Algorithms

From Absolute RT to Relative RT
Approximating RT Functions
Asymptotic Upper Bound (Big-O): Def.

Announcements/Reminders

- Assignment 1 released
- ***splitArrayHarder***: an **extended** version released
- Office Hours: 3pm to 4pm, Mon/Tue/Wed/Thu
- Contact Information of TAs on common eClass site

Example 2: Counting Number of Primitive Operations

(Exercise)

```
1  boolean foundEmptyString = false;
2  int i = 0;
3  while (!foundEmptyString && i < names.length) {
4    if (names[i].length() = 0) {
5      /* set flag for early exit */
6      foundEmptyString = true;
7    }
8    i = i + 1;
9  }
```

Q. # of times Line 3 is executed?

Q. # of times loop body (Lines 4 to 8) is executed?

Q. # of POs in the loop body (Lines 4 to 8)?

Comparing Algorithms: From Absolute RT to Relative RT

t
exact time taken to execute a PO in some environment
exec.

e.g. Mac M1 t = 2 (ms)
e.g. Mac'14 t = 4 (ms)

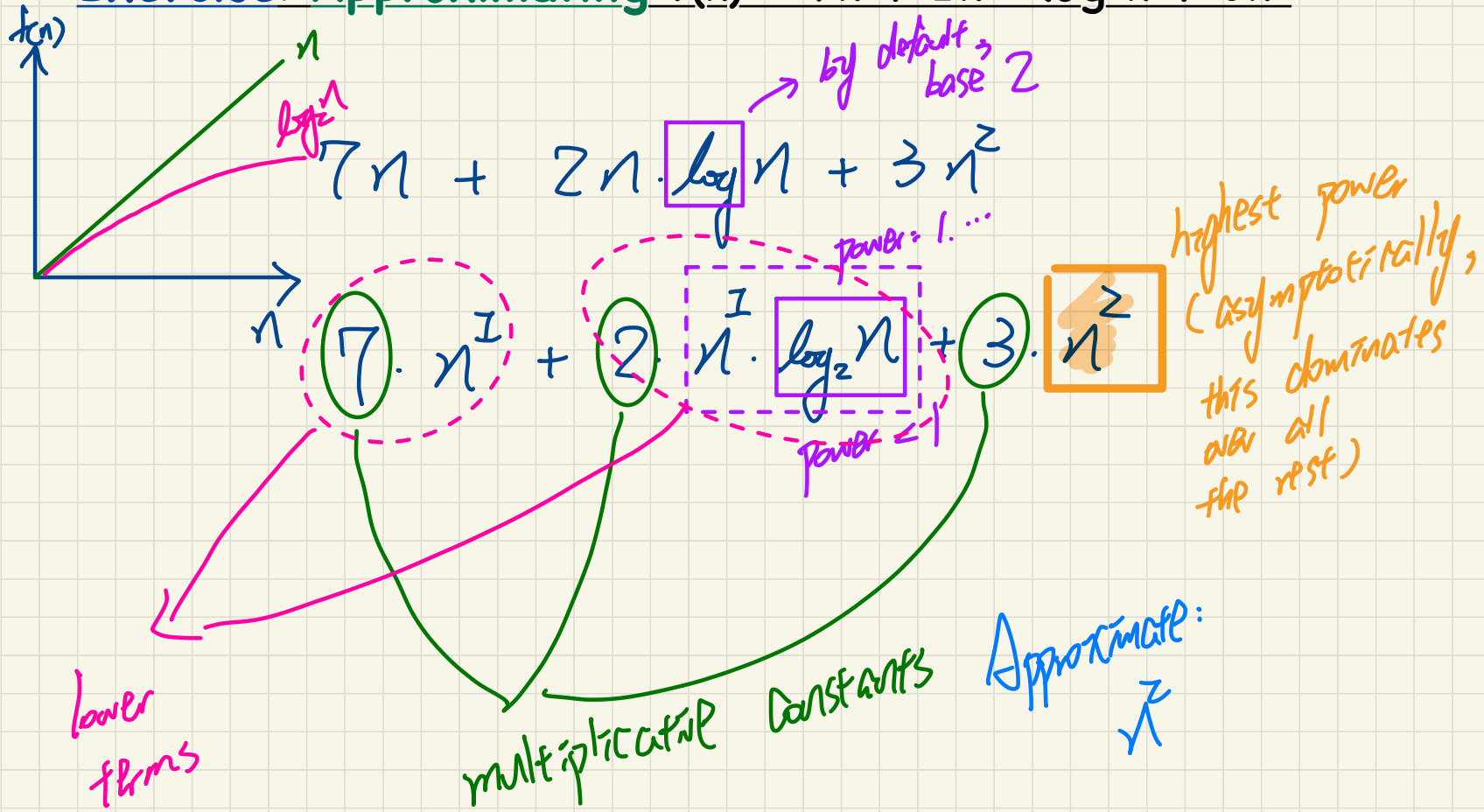
abs. RT	Mac M1	Mac'14
100-2	(100·1-2)·2	(100·1-2)·4
10n+3	(100·10+3) 2	

input size
assume n=100

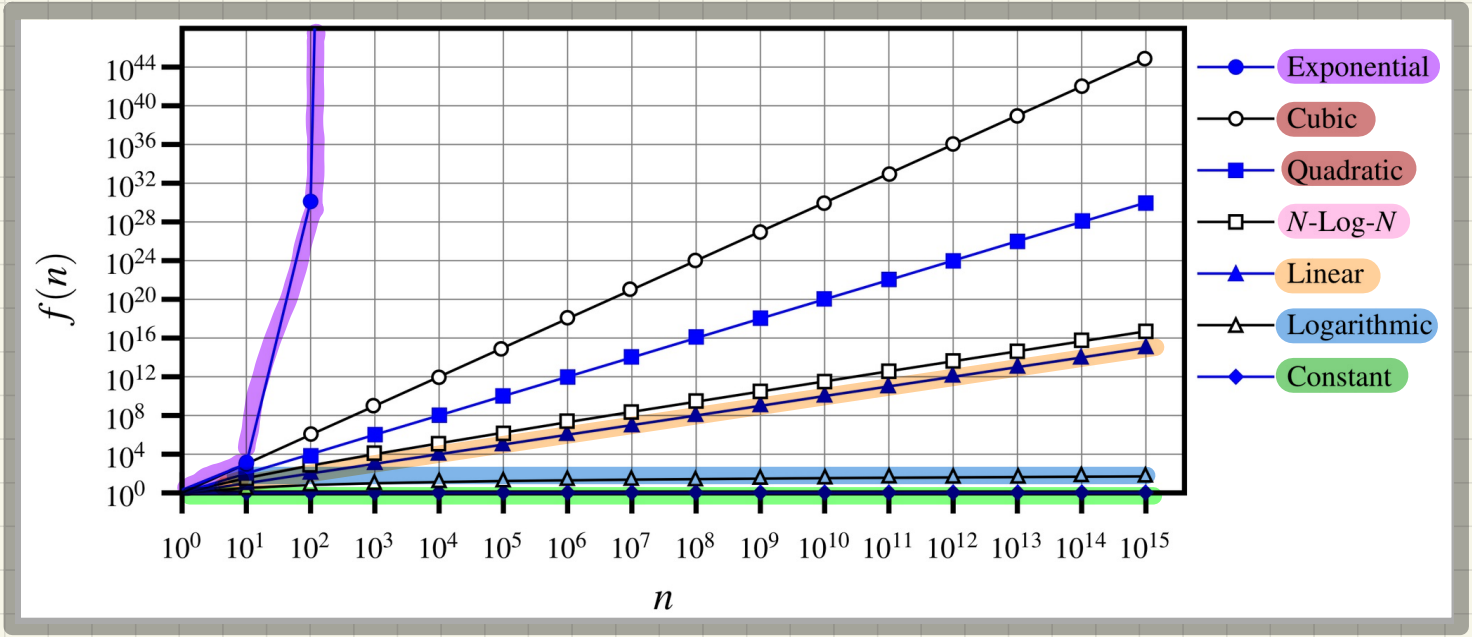
for the same alg., t is not useful (∵ we're supposed to compare it with another alg.).

to compare the RT of two alg. : not necessary → same conclusion can be drawn anyway.
to know the precise value of t (e.g. 2, 4)

Exercise: Approximating $f(n) = 7n + 2n \cdot \log n + 3n^2$



RT Functions: Rates of Growth (w.r.t. Input Sizes)



RT $f(n) = 1000$ $f(1M) = 1000$

$f(1) = 1000$ $f(1B) = 1000$

$f(100) = 1000$

Comparing Relative, Asymptotic RTs of Algorithms

Q1. Compare:

$RT_1(n) = 3n^2 + 7n + 18 \approx n^2$

$RT_2(n) = 100n^2 + 3n - 100 \approx n^2$

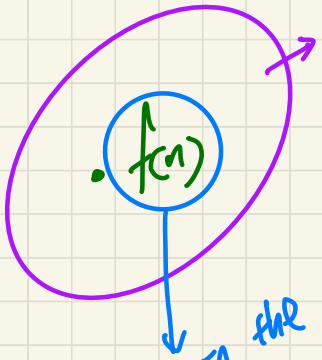
↳ equally efficient, asymptotically.

Q2: Compare:

$RT_1(n) = n^3 + 7n + 18 \approx n^3$

$RT_2(n) = 100n^2 + 100n + 2000 \approx n^2$

↳ RT_2 more efficient (taking less time) asymptotically.



$f(n)$: RT function

↳ input size ⇝ relative RT

$O(g(n))$

e.g. $f(n) = 7n - 2$

$g(n)$: reference function → $g(n) \rightsquigarrow c \cdot g(n)$ (modifies the slope of $g(n)$)

(further manipulation on $g(n)$ expected)

$f(n)$ being in the family means that it can somehow be upper-bounded by $g(n)$

Goal: Prove $f(n)$ is $O(g(n))$

Asymptotic Upper Bound: Big-O

upper bound effect.

$f(n) \in O(g(n))$ if there are:

- A real *constant* $c > 0$
- An integer *constant* $n_0 \geq 1$

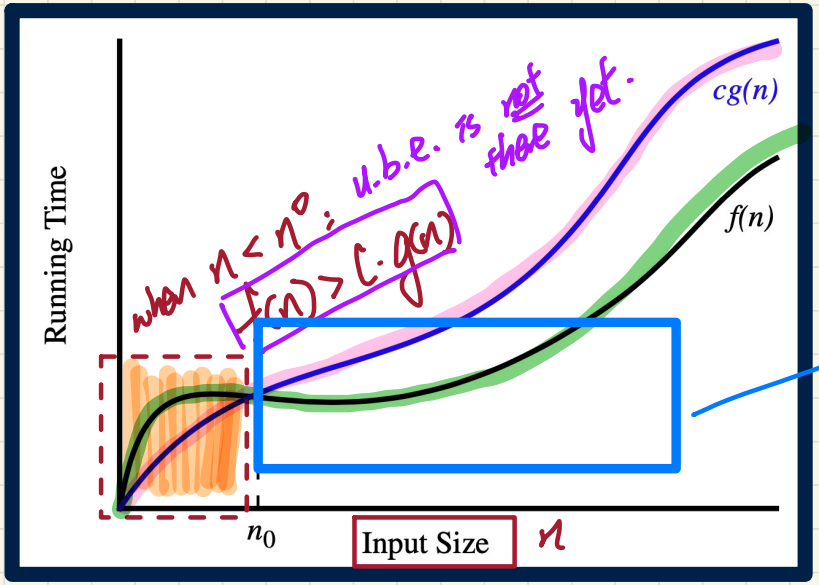
such that:

$$f(n) \leq c \cdot g(n) \quad \text{for } n \geq n_0$$

multiplicative constant applied to g(n) to change its slope

starting point of the u.b.e.

u.b.e.



Example:

f(n) = 8n + 5

g(n) = n

Prove:

f(n) is O(g(n))

Choose c = 9.

What about n0?

starting from $n = n_0$,

$f(n) \leq c \cdot g(n)$

upper bound effect is there!

Can n0 be 1?

$f(1) \leq 9 \cdot 1$

13 ↓ False

no u.b.e yet.

$* \; 1^0 = 1^2 = \cdots = 1^d = 1$

Proving f(n) is O(g(n))

We prove by choosing

$$\begin{array}{|l} c \\ n_0 \end{array} \; = \; \begin{array}{l} |a_0| + |a_1| + \cdots + |a_d| \\ \hline 1 \end{array}$$

If $f(n)$ is a polynomial of degree d, i.e.,

$$f(n) = a_0 \cdot n^0 + a_1 \cdot n^1 + \cdots + a_d \cdot n^d$$

and $a_0, a_1, \ldots, a_d$ are integers (i.e., negative, zero, or positive),

then $f(n)$ is $O(n^d)$. → $g(n)$

$** \; a_0 \le |a_0|$, $a_1 \le |a_1|$, ⋮ , $a_d \le |a_d|$

(1) $f(1) \le c \cdot 1^d$

(2) $f(n) \le c \cdot n^d$ $(n > 1)$

Upper-bound effect: $n_0 = 1$? $\quad [f(1) \le (|a_0| + |a_1| + \cdots + |a_d|) \cdot 1^d]$

$$f(1) = a_0 \cdot 1^0 + a_1 \cdot 1^1 + \cdots + a_d \cdot 1^d$$

$$\overset{*}{=} (a_0 + a_1 + \cdots + a_d) \cdot 1^d \overset{**}{\le} (|a_0| + |a_1| + \cdots + |a_d|) \cdot 1^d$$

Upper-bound effect holds? $\quad [f(n) \le (|a_0| + |a_1| + \cdots + |a_d|) \cdot n^d]$

Lecture 6 - January 23

Asymptotic Analysis of Algorithms

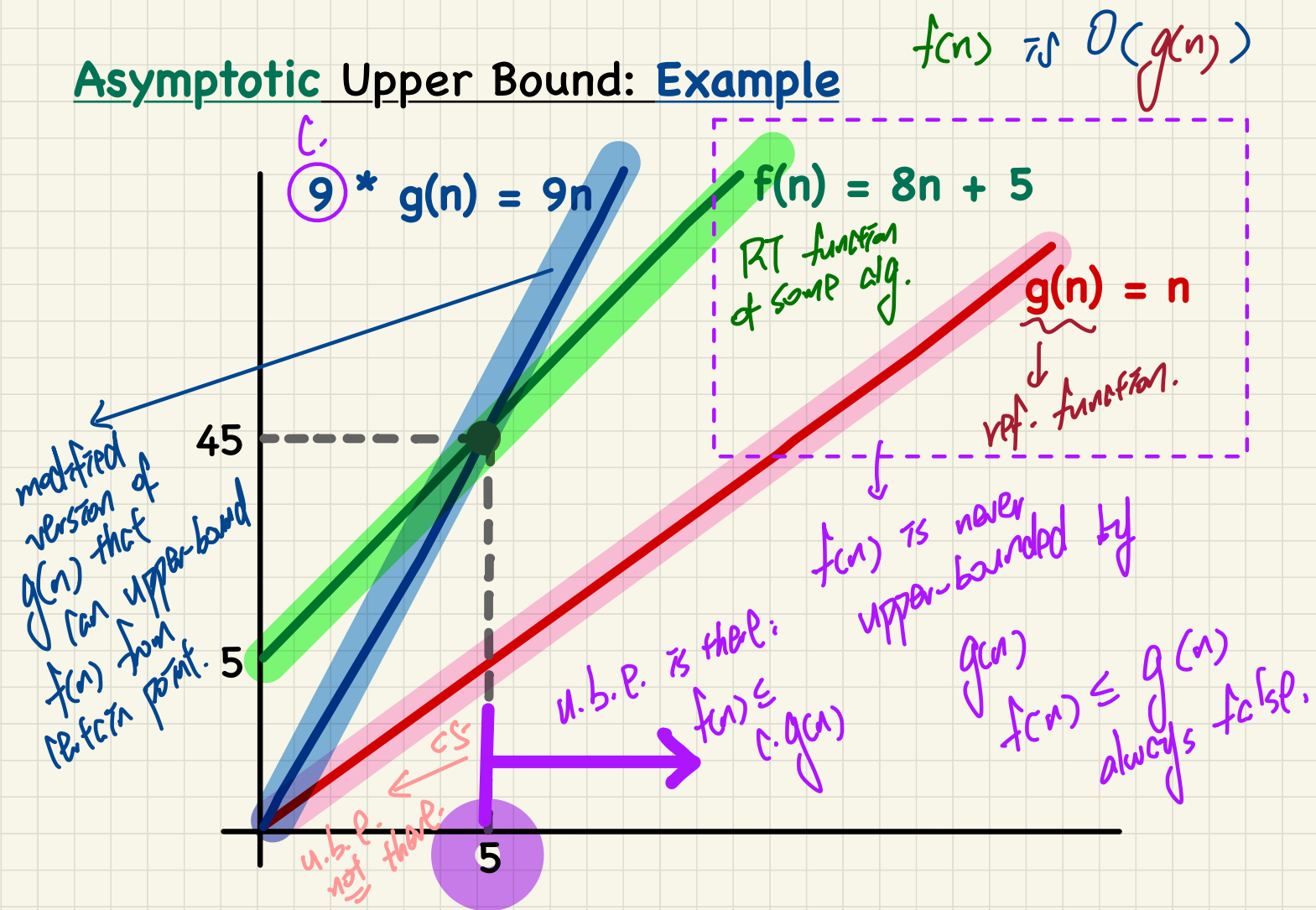
Big-O: Pred. Def., Properties, Examples
Correct vs. Accurate Asymptotic U.B.
Deriving U.B. from Code: Basic Examples

Announcements/Reminders

- **Assignment 1** due next Monday
- ***splitArrayHarder***: an **extended** version released
- Office Hours: 3pm to 4pm, Mon/Tue/Wed/Thu
- Contact Information of TAs on common eClass site

Asymptotic Upper Bound: Example

$f(n)$ is $O(g(n))$



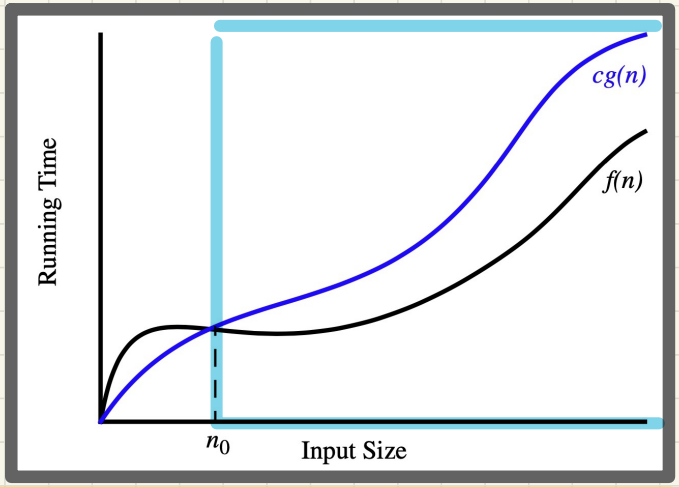
Asymptotic Upper Bound (Big-O): Alternative Formulation

Known:

$f(n) \in O(g(n))$ if there are:
- A real *constant* $c > 0$
- An integer *constant* $n_0 \geq 1$

such that:

$$f(n) \leq c \cdot g(n) \quad \text{for } n \geq n_0$$



Q. Formulate the definition of "f(n) is order of O(g(n))" using **logical** operator(s): $\neg, \wedge, \vee, \Rightarrow, \forall, \exists$

$$f(n) \in O(g(n)) \iff \exists c, n_0 \cdot (c > 0 \wedge n_0 \geq 1) \wedge (\forall n \cdot n \geq n_0 \Rightarrow f(n) \leq c \cdot g(n))$$

$* \; 1^0 = 1^1 = \cdots = 1^d = 1$

Proving f(n) is O(g(n))

We prove by choosing

$$c = |a_0| + |a_1| + \cdots + |a_d|$$
$$n_0 = 1$$

If $f(n)$ is a polynomial of degree d, i.e.,

$$f(n) = a_0 \cdot n^0 + a_1 \cdot n^1 + \cdots + a_d \cdot n^d$$

and $a_0, a_1, \ldots, a_d$ are integers (i.e., negative, zero, or positive), then $f(n)$ is $O(n^d)$. → $g(n)$

(1) $f(1) \le c \cdot 1^d$

(2) $f(n) \le c \cdot n^d$ $(n > 1)$

$a_0 \le |a_0|$, $a_1 \le |a_1|$, $\vdots$, $a_d \le |a_d|$

Upper-bound effect: $n_0 = 1$? $[f(1) \le (|a_0| + |a_1| + \cdots + |a_d|) \cdot 1^d]$

$$f(1) = a_0 \cdot 1^0 + a_1 \cdot 1^1 + \cdots + a_d \cdot 1^d$$
$$\overset{*}{=} (a_0 + a_1 + \cdots + a_d) \cdot 1^d \overset{**}{\le} (|a_0| + |a_1| + \cdots + |a_d|) \cdot 1^d$$

$1^0 \le n^d$, $1^1 \le n^d$, $\vdots$, $1^d \le n^d$

Upper-bound effect holds? $[f(n) \le (|a_0| + |a_1| + \cdots + |a_d|) \cdot n^d]$ $n > 1$

$$f(n) = a_0 \cdot n^0 + a_1 \cdot n^1 + \cdots + a_d \cdot n^d$$
$$\overset{***}{\le} (a_0 + a_1 + \cdots + a_d) \cdot n^d \overset{**}{\le} (|a_0| + |a_1| + \cdots + |a_d|) \cdot n^d$$

$n^0 \le n^d$, $n^1 \le n^d$

Exercise: Prove $f(n) = 5n^4 - 3n^3 + 2n^2 - 4n + 1$ **is** $O(n^4)$

$g(n)$

highest power

$f(n) = 5n^4 - 3n^3 + 2n^2 - 4n + 1 \cdot n^0$

(1) Parse/Guess: $f(n)$ is $O(n^4)$

(2) Prove:

choose c: $|5| + |(-3)| + |2| + |(-4)| + |1| = 15$

No: 1

Verify

$f(1) \leq c \cdot g(1)$

$5 - 3 + 2 - 4 + 1 \leq 15 \cdot 1$

True

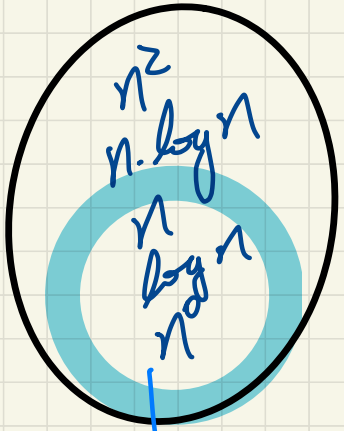
Big-O Properties (1): Members in a Family

Each member f(n) in O(g(n)) is such that:

Higest Power of f(n) <= Highest Power of g(n)

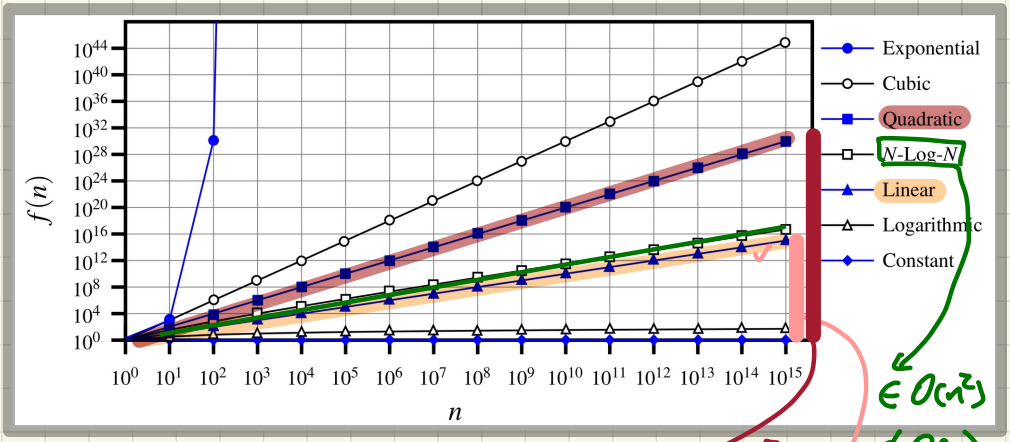
O(n)

O(n²)



O(n) "is contained" within O(n²)

Functions: Rates of Growth



∈ O(n²)

∉ O(n)

O(n³) contains O(n) plus other functions growing not as fast as n²

O(n) contains all functions growing no faster than n.

Big-O Properties (2): Relating Families

$O(n)$? $O(n^2)$

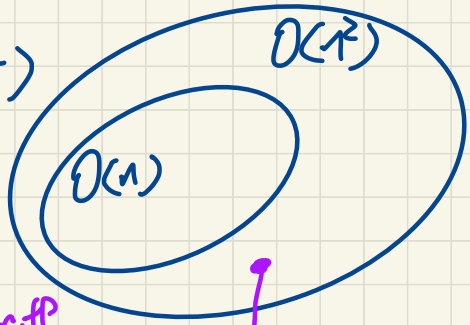
(1) ~~$\subseteq$~~

Correct (2) $\subset$

(3) $\supseteq$

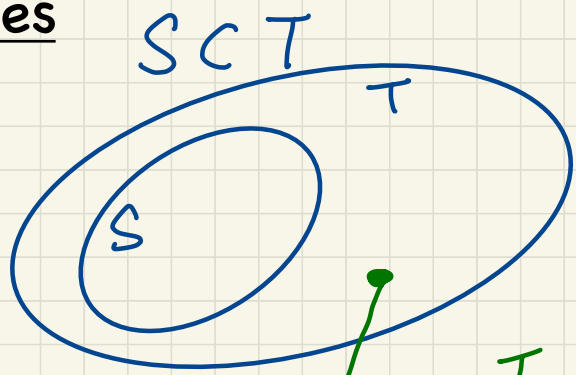
(4) $\supset$

more accurate.



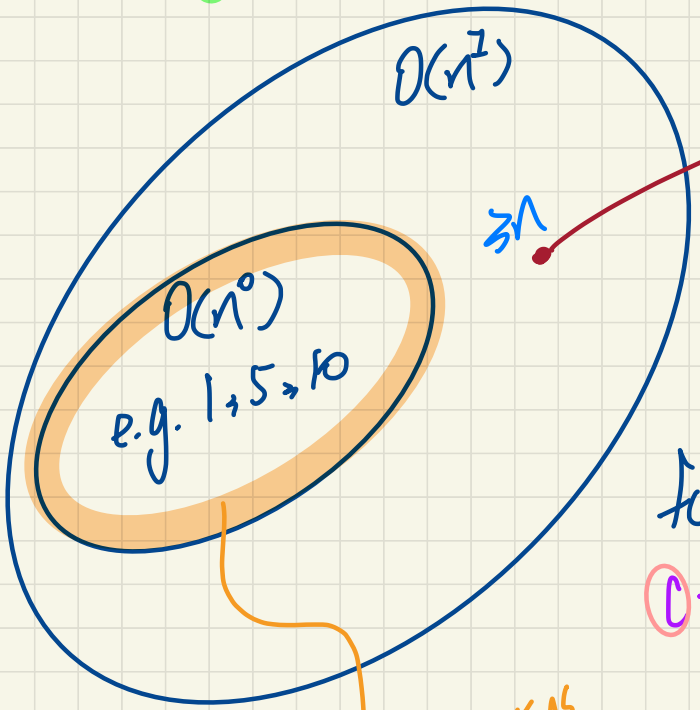
a function that can be u.b. by n^2 but cannot be u.b. by n

e.g. n^2, $2 \cdot n^2$, $n \cdot \log n$



a member in T but not in S

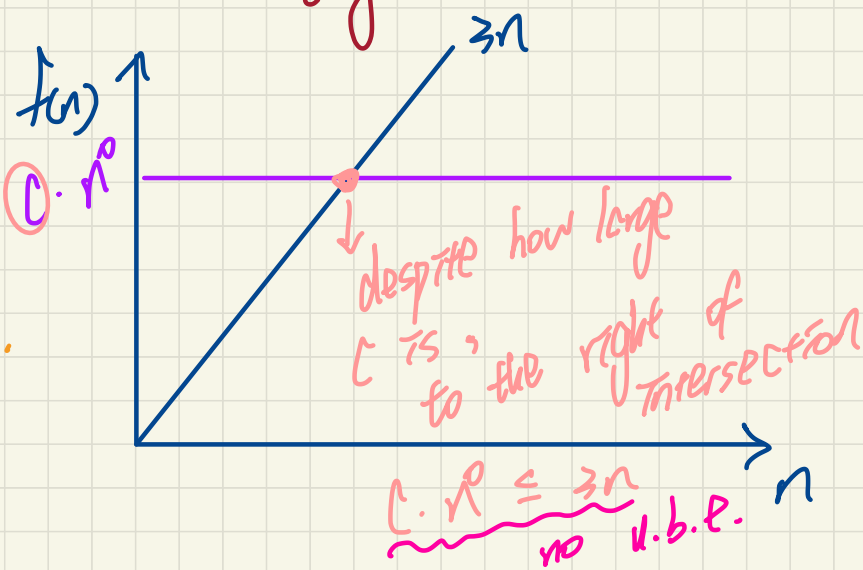
$O(n^0) \subset O(n)$



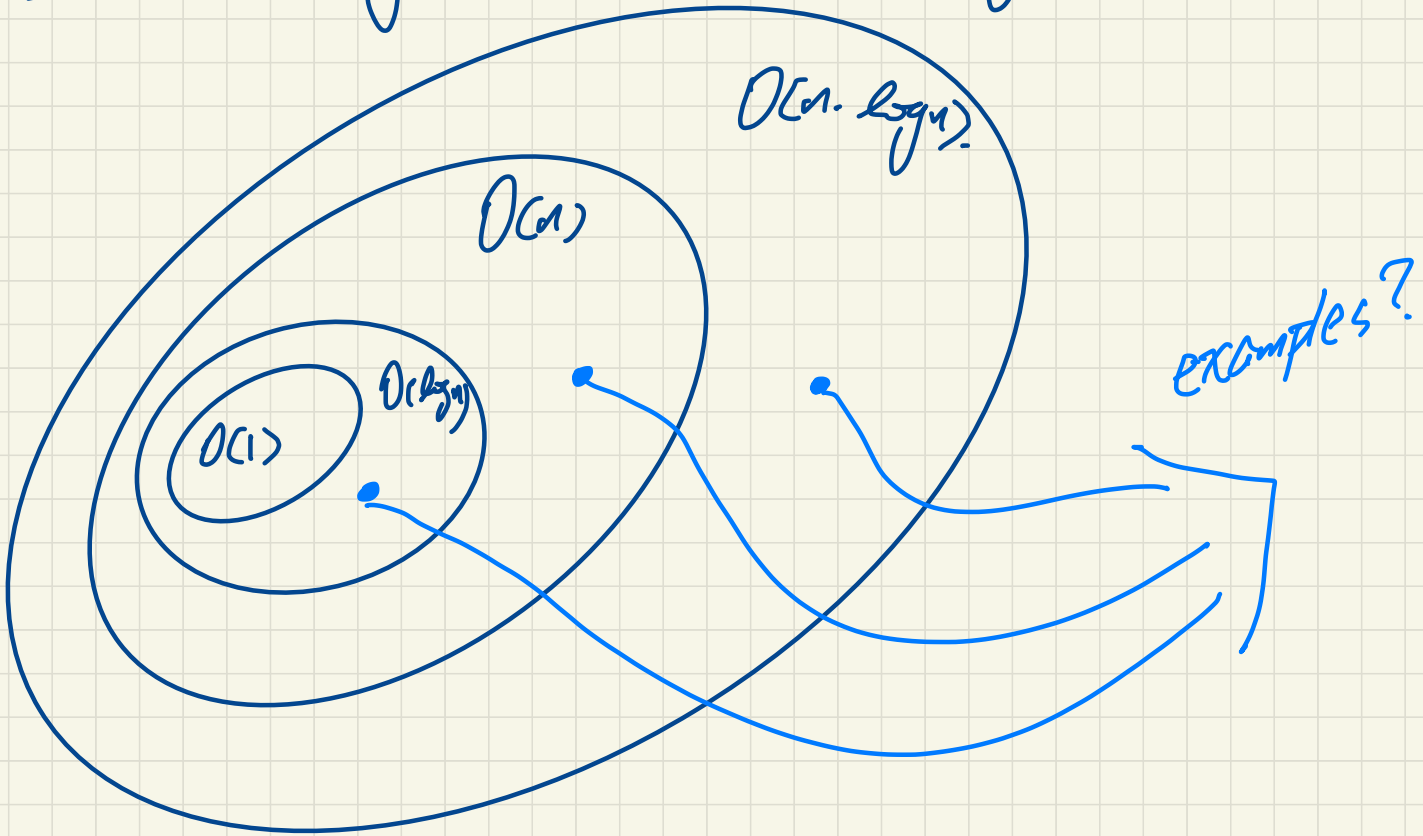
set of functions that can be u.b. by n^0

at least one function that can be u.b. by n^1 but cannot be u.b. by n^0

e.g. $f(n) = 3n$



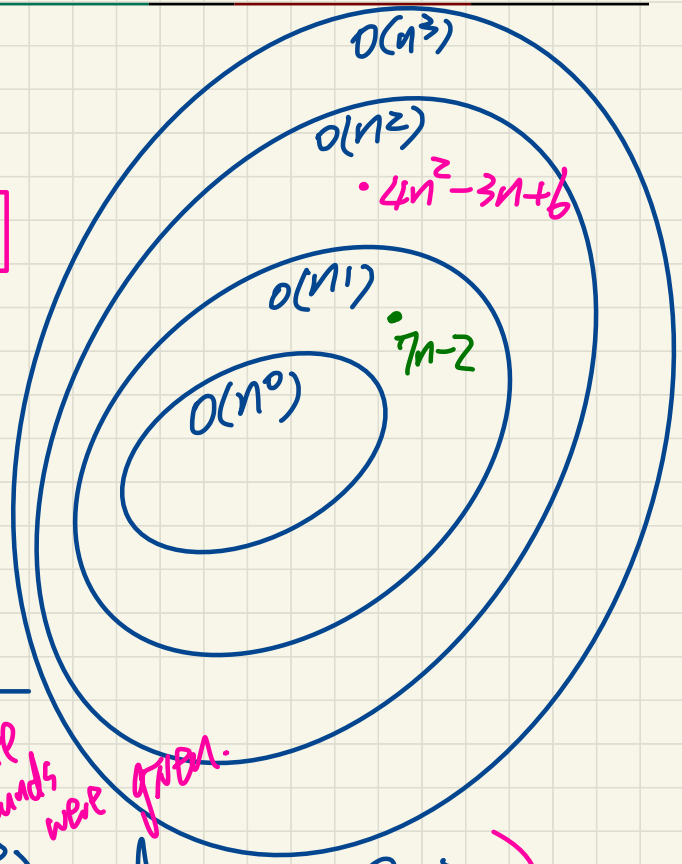
$O(n^0) \subset O(\log n) \subset O(n) \subset O(n \cdot \log n) \subset \cdots$



Big-O Properties (3): Deciding Correct & Accurate Bound

e.g. $f_1(n) = 7n - 2$

$f_2(n) = 4n^2 - 3n + b$



$7n - 2$ is order of ($\in$)

- $O(n)$
- $O(n^2)$
- $O(n^3)$
- $O(2^n)$

Correct.

smallest power that can upper-bound $7n - 2$ ⇒ most accurate u.b.

Misleading !! not accurate bounds were given.

$f_1(n)$ is $O(n^3)$; $f_2(n)$ is $O(n^2)$

$O(7n^2 + 4n - 2)$ ✗ → always exclude lower terms and mul. constants.

$O(n^2)$ ✓

Asymptotic Upper Bounds: Example (1)

Given $f(n) = 5n^2 + 3n \cdot \log n + 2n + 5$:

(1) What is f(n)'s **most accurate** asymptotic upper bound.

(2) **Prove** your claim.

(1) $O(n^2)$

(2) Choose $c: |5| + |3| + |2| + |5| = 15$.

$n_0: 1$ → verify $f(1) \le 15 \cdot 1^2$

↓ u.b.e. stands at $n_0 = 1$.

Asymptotic Upper Bounds: Example (2) (Exercise)

Given $f(n) = 20n^3 + 10n \cdot \log n + 5$:

(1) What is f(n)'s **most accurate** asymptotic upper bound.

(2) **Prove** your claim.

Asymptotic Upper Bounds: Example (3)

Given $f(n) = 3 \cdot \log n + 2$:

(1) What is f(n)'s **most accurate** asymptotic upper bound.

(2) **Prove** your claim.

(1) $O(\log n)$ $g(n)$

(2) Choose: $C = |3| + |2| = 5$.

$n_0 = 1$ ✗ → Verify: $f(1) \leq 5 \cdot \log 1$

$0 \neq 2^0 = 1$

what about $n_0 = 2$

↳ $f(2) \leq 5 \cdot \log 2$ (Exercise)

↓ u.b.e.?

$3 \cdot \log 1 + 2$

$2 \leq 0$

u.b.e. not there.

Asymptotic Upper Bounds: Example (4) (Exercise)

Given $f(n) = 2^{n+2}$:

(1) What is f(n)'s **most accurate** asymptotic upper bound.

(2) **Prove** your claim.

Asymptotic Upper Bounds: Example (5) (Exercise)

Given $f(n) = 2n + 100 \cdot \log n$:

(1) What is f(n)'s **most accurate** asymptotic upper bound.

(2) **Prove** your claim.

Determining the Asymptotic Upper Bound (1)

```
1  int maxOf (int x, int y) {
2    int max = x;
3    if (y > x) {
4      max = y;
5    }
6    return max;
7  }
```

1
1
1
1

O(1+1+1+1)

= O(4)

4 ∈ n^0

O(1)

Determining the Asymptotic Upper Bound (2)

```
1  int findMax (int[] a, int n) {
2    currentMax = a[0];
3    for (int i = 1; i < n; ) {
4      if (a[i] > currentMax) {
5        currentMax = a[i];
6      i ++; }
7    return currentMax; }
```

$O(1+n+n+n+n+1)$

$O(4n+2)$

$O(n)$

Lecture 7 - January 28

Asymptotic Analysis of Algorithms, Arrays and Linked Lists

Deriving Upper Bounds from Code
Inserting into an Array
Sorting Orders

Announcements/Reminders

- **Assignment 1** solution to be released **soon**
- ***splitArrayHarder***: an **extended** version released
- Office Hours: 3pm to 4pm, Mon/Tue/Wed/Thu
- Contact Information of TAs on common eClass site

$[a, b]$

↳ # of integers in this closed interval is

$$b - a + 1$$

$[0, n-1]$

↳ $(n-1) - 0 + 1 = n$

$[2, 99]$

↳ $99 - 2 + 1$

Determining the Asymptotic Upper Bound (3.1)

```
1  boolean containsDuplicate (int[] a, int n) {
2    for (int i = 0; i < n; ) {
3      for (int j = 0; j < n; ) {
4        if (i != j && a[i] == a[j]) {
5          return true; }
6        j ++; }
7      i ++; }
8    return false; }
```

P1 (lines 4–6) O(1); P2 (line 7) O(1); line 8 O(1)

* P1 gets executed for each combination of (i, j)
* P2 gets executed for each value of i.

Pattern of Loop Counters

outer loop runs $[0, n-1]$ — n iterations

i	j					
0	0	1	2	⋯	n-1	n iterations
1	0	1	2	⋯	n-1	
2	0	1	2	⋯	n-1	
⋮			⋮			
n-1	0	1	2	⋯	n-1	

$(i, j) = (2, 2)$ → when L4~L6 exec. once.

$$O(\underbrace{n}_{\#\text{ values of } i} \cdot \underbrace{n}_{\#\text{ values of } j \text{ for each } i} \cdot \underbrace{1}_{L4\text{-}L6} + \underbrace{n}_{\#\text{ values of } i} \cdot \underbrace{1}_{L7} + \underbrace{1}_{L8})$$

$$= O(n^2 + n + 1)$$

$$= O(n^2)$$

Determining the Asymptotic Upper Bound (3.2)

```
1  boolean containsDuplicate (int[] a, int n) {
2    for (int i = 0; i < n; ) {
3      for (int j = 0; j < n; ) {
4        if (i != j && a[i] == a[j]) {
5          return true; }
6        j ++; }
7      i ++; }
8    return false; }
```

$O(10{,}000 \cdot n)$

$= O(n)$

```
1  boolean containsDuplicate (int[] a, int n) {
2    for (int i = 0; i < n; ) {
3      for (int j = 0; j < n; ) {
4        if (i != j && a[i] == a[j]) {
5          return true; }
6        j ++; }
7      i ++; }
8    return false; }
```

$O(1M \cdot 1M)$

$= O(c \cdot n^0)$

$= O(1)$

Determining the Asymptotic Upper Bound (4)

```
1  int sumMaxAndCrossProducts (int[] a, int n) {
2    int max = a[0];
3    for(int i = 1; i < n; i ++) {
4      if (a[i] > max) { max = a[i]; }
5    }
6    int sum = max;
7    for (int j = 0; j < n; j ++) {
8      for (int k = 0; k < n; k ++) {
9        sum += a[j] * a[k]; } }
10   return sum; }
```

1

n

1

n^2

1

$O(1+n+1+n^2+1)$

$= O(n^2+2n+3)$

$= O(n^2)$

Determining the Asymptotic Upper Bound (5)

```
1  int triangularSum (int[] a, int n) {
2    int sum = 0;
3    for (int i = 0; i < n; i ++) {
4      for (int j = i; j < n; j ++) {
5        sum += a[j]; } }
6    return sum; }
```

1 (line 2) · 1 (line 5) → exec for each (i, j) · 1 (line 6)

Pattern of (i, j)

# of times L5 is exec: $n + (n-1) + (n-2) + \cdots + 1$

$= \frac{(n+1) \cdot n}{2}$ is $O(n^2)$

# [0, n-1]	i	j					
n	0	0	1	2	⋯	n-1	[0, n-1] → n
	1		1	2	⋯	n-1	[1, n-1] → n-1
	2			2	⋯	n-1	[2, n-1] → n-2
	⋮			⋮		⋮	
	i						
	n-1					n-1	[n-1, n-1] → 1

$O(\underbrace{1}_{L2} + \underbrace{n^2}_{\#(i,j)} \cdot \underbrace{1}_{L5} + \underbrace{1}_{L6})$

$= O(n^2 + 2) = O(n^2)$

Asymptotic Upper Bound: Arithmetic Sequence/Progression

$$\underbrace{i + 0 \cdot c}_{\text{start term}} + \overbrace{(i + c)}^{\text{common difference}} + (i + 2 \cdot c) + \cdots + \underbrace{(i + (n-1) \cdot c)}_{n \text{ terms}}$$

$$\text{Sum} \quad \frac{(i + (i + (n-1) \cdot c)) \cdot n}{2} = \frac{c \cdot n^2 + (2i - c) \cdot n}{2}$$

is order of $O(n^2)$

Inserting into an Array

object creation: O(1)

[i+1, n] → n-(i+1)+1

insert "e" to index "i" (0 ~ n)

a.length

```
String[] insertAt(String[] a, int n, String e, int i)
1  String[] result = new String[n + 1];                    1
n* for(int j = 0; j <= i - 1; j ++){ result[j] = a[j]; }   j: [0, i-1] → i iterations → n times
1  result[i] = e;
n  for(int j = i + 1; j <= n; j ++){ result[j] = a[j-1]; }  j: [i+1, n] → n-i → n times
1  return result;
```

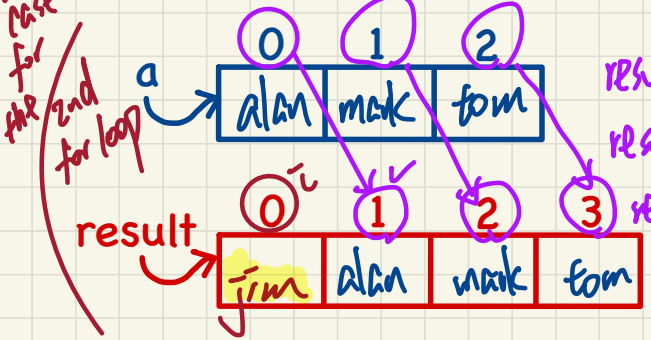
* O(n · 1) → max value of i

O(1+n+1+n+1) = O(n)

Example:

insertAt({alan, mark, tom}, 3, jim, 0)

worst case for the 2nd for loop



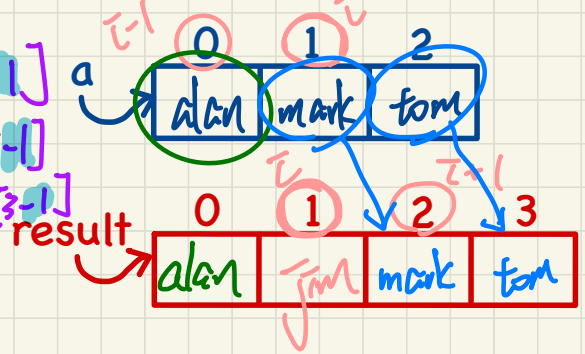
result[1] == a[1-1]

result[2] == a[2-1]

result[3] == a[3-1]

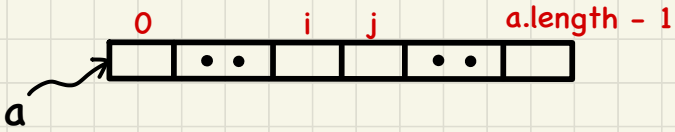
Example:

insertAt({alan, mark, tom}, 3, jim, 1)



Exercise: insertAt({alan, mark, tom}, 3, jim, 3)

Sorting Orders of Arrays

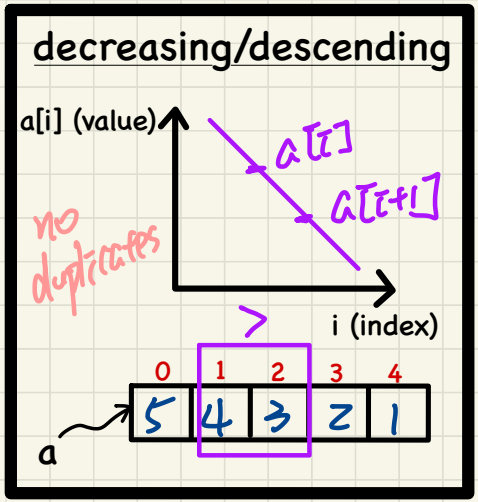


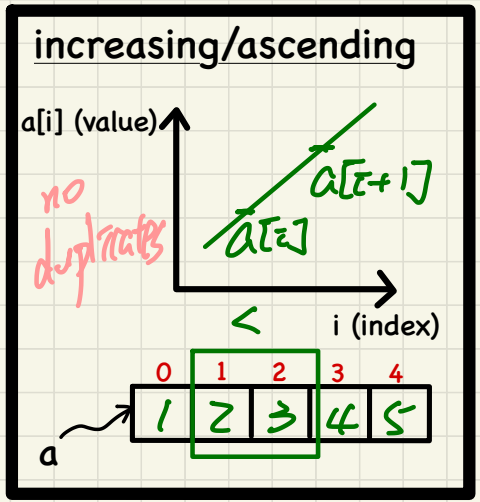
non-descending

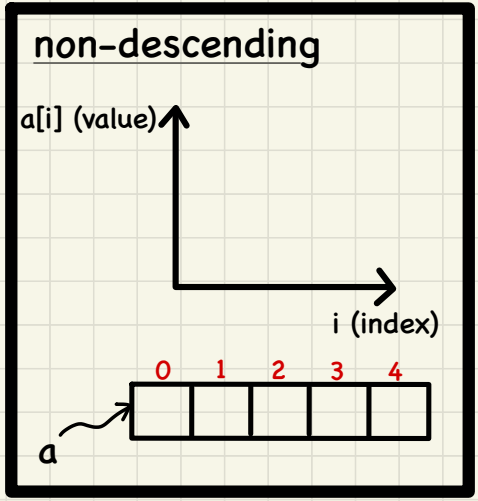
$\triangleq \neg(\text{descending})$

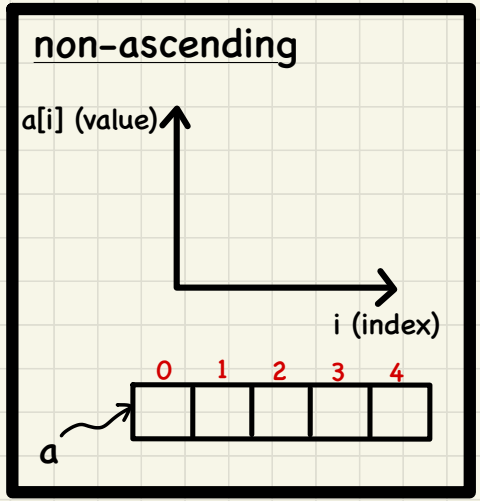
$\equiv \neg(a[i] > a[i+1])$

$\equiv a[i] \leq a[i+1]$









Lecture 8 - January 30

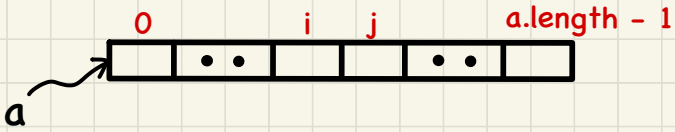
Arrays and Linked Lists

Exercise: Relating Sorting Orders
Selection vs. Insertion Sorts
Singly-Linked List: Quick, Visual Intro.

Announcements/Reminders

- Assignment 1 solution released
- ***splitArrayHarder***: an **extended** version released
- Lecture notes template available
- Office Hours: 3pm to 4pm, Mon/Tue/Wed/Thu
- Contact Information of TAs on common eClass site

Sorting Orders of Arrays

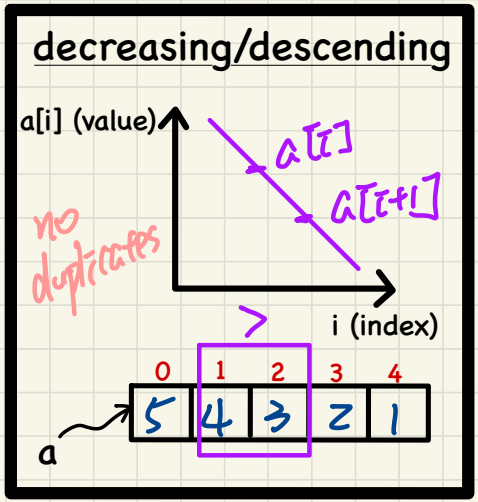


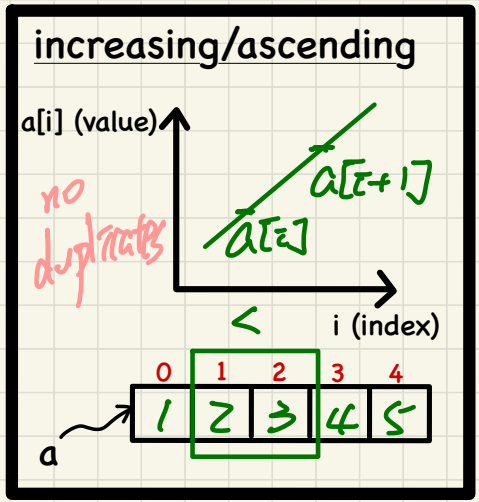
non-descending

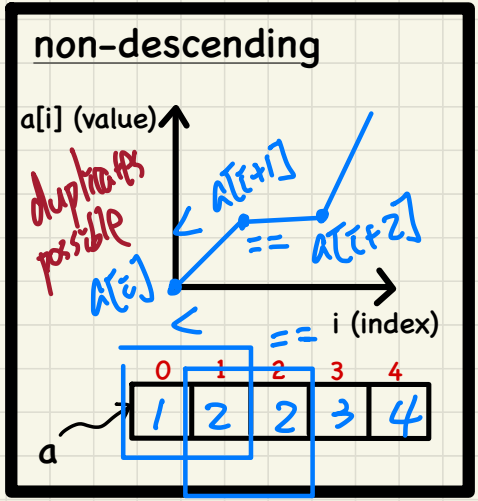
$\triangleq \neg$ (descending)

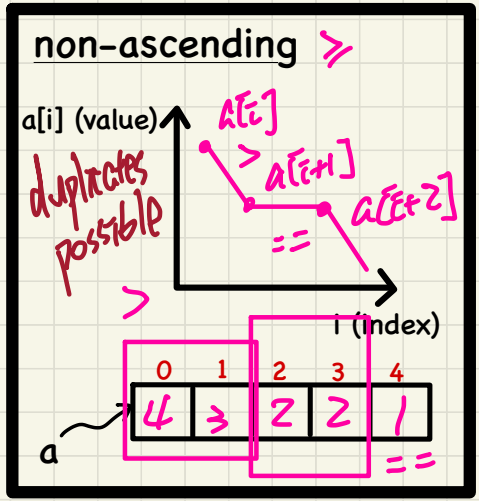
$\equiv \neg (a[i] > a[i+1])$

$\equiv a[i] \leq a[i+1]$









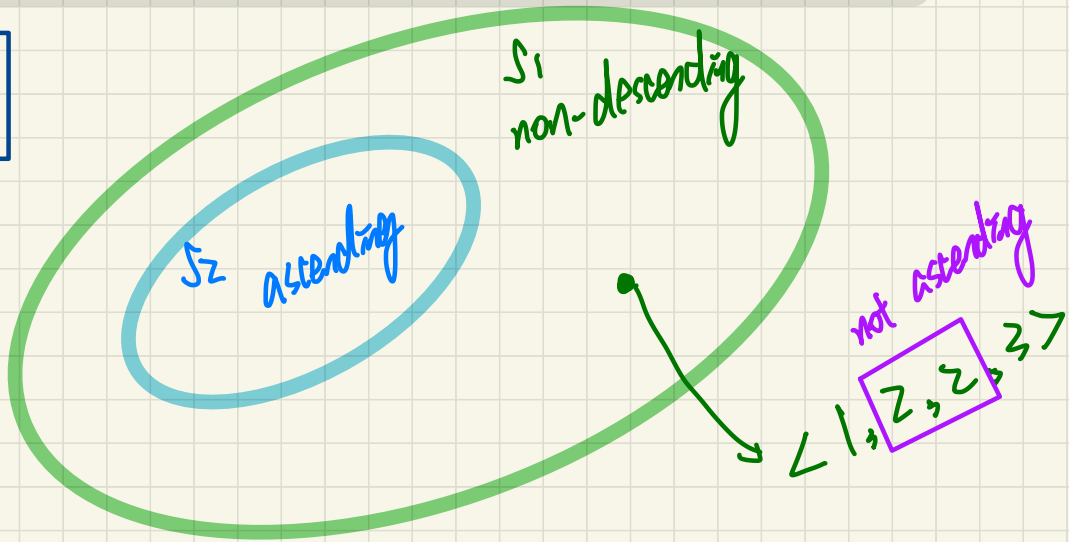
Exercise: Relating Sets of Sorted Arrays

Q. Consider the following two sets:

- S1: all arrays sorted in a **non-descending** order
- S2: all arrays sorted in an **ascending** order.

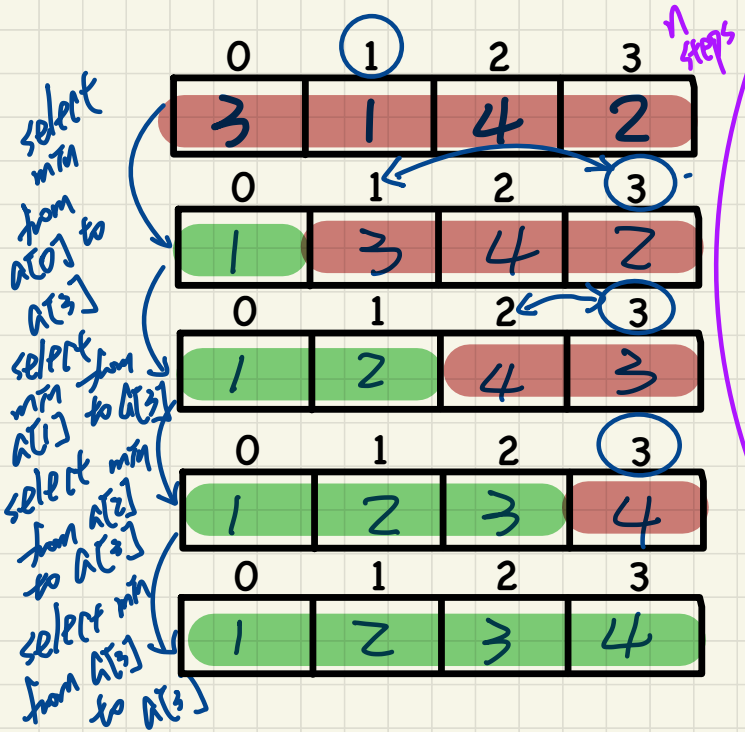
Formulate the relation between these two sets.

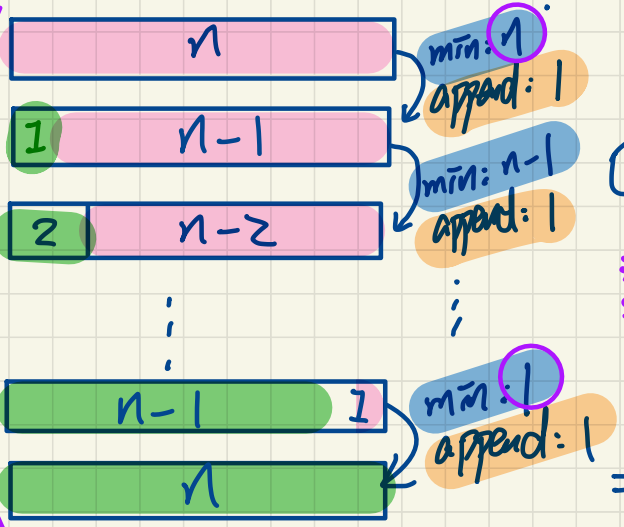
$S_2 \subset S_1$



Selection Sort

Keep **selecting** minimum from the **unsorted** portion and appending it to the end of **sorted** portion.





$$O(\underbrace{n}_{\#\text{steps}} \cdot \underbrace{1}_{\text{swap}} + [\overbrace{n}^{\text{1st selec.}} + (n-1) + \dots + \underbrace{1}_{\text{last selec.}}])$$

$$= O(n + n^2) = O(n^2)$$

Insertion Sort

$O(1)$

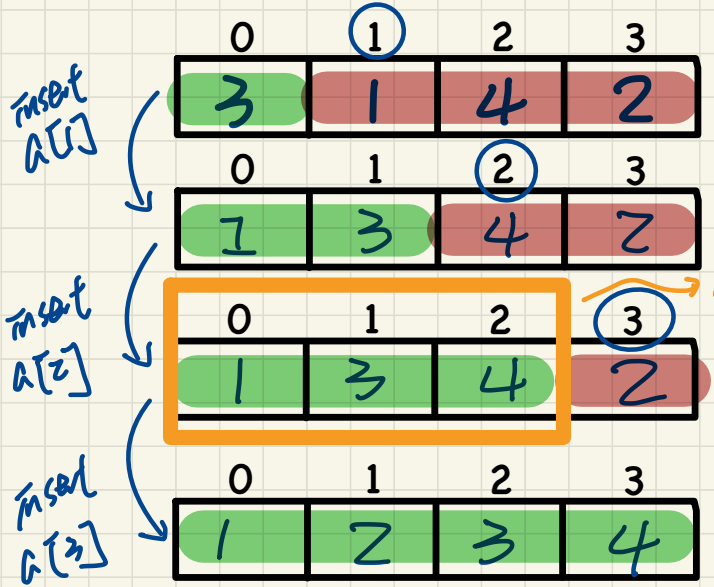
$O\left((n-1)\cdot 1 + [1+2+3+\cdots+(n-1)]\right)$

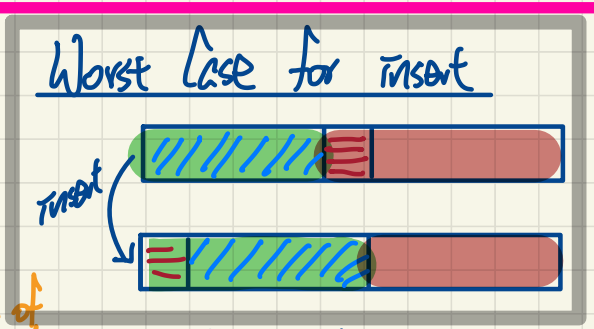
steps · choosing 1st unsorted item

$= O(n + n^2)$

$= O(n^2)$

Keep getting 1st element from the **unsorted** portion and **inserting** it to the **sorted** portion.





Step	choose	size of sorted portion	for insert max cost
1	a[1]	1	1
2	a[2]	2	2
3	a[3]	3	3
⋮	⋮	⋮	⋮
n-1	a[n-1]	n-1	n-1

Selection Sort: Deriving Asymptotic Upper Bound

```
1  void selectionSort(int[] a, int n)
2    for (int i = 0; i <= (n - 2); i ++)
3      int minIndex = i;
4      for (int j = i; j <= (n - 1); j ++)
5        if (a[j] < a[minIndex]) { minIndex = j; }
6      int temp = a[i];
7      a[i] = a[minIndex];
8      a[minIndex] = temp;
```

a.length

1

1

1

exec. according to values of i

exec. acc. to values (i, j)

i: 0, 1, 2, ..., n-2 → [0, n-2] → n-1 iterations

i	j	
0	1 2 ... n-1	n
1	1 2 ... n-1	n-1
2	2 ... n-1	n-2
⋮	⋮	⋮
n-2	n-1	2

$$O\left((n-1)\cdot 1 + \underbrace{[n+(n-1)+\cdots+2]}_{\#(i,j)}\cdot 1\right)$$

(n-1): # it. of outer loop; 1: Lb-Ub

$$\frac{(n+2)\cdot(n-1)}{2} \approx n^2$$

$$= O(n-1+n^2)$$

$$= O(n^2)$$

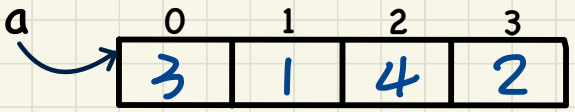
Insertion Sort: Deriving Asymptotic Upper Bound

```
1  void insertionSort(int[] a, int n)
2    for (int i = 1; i < n; i ++)
3      int current = a[i];
4      int j = i;
5      while (j > 0 && a[j - 1] > current)
6        a[j] = a[j - 1];
7        j --;
8      a[j] = current;
```

Exercise

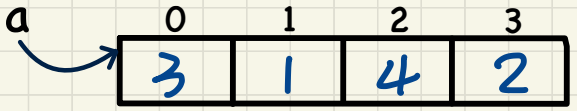
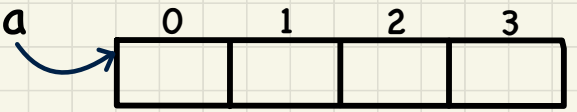
Selection Sort in Java

```
1  void selectionSort(int[] a, int n)
2    for (int i = 0; i <= (n - 2); i ++)
3      int minIndex = i;
4      for (int j = i; j <= (n - 1); j ++)
5        if (a[j] < a[minIndex]) { minIndex = j; }
6      int temp = a[i];
7      a[i] = a[minIndex];
8      a[minIndex] = temp;
```



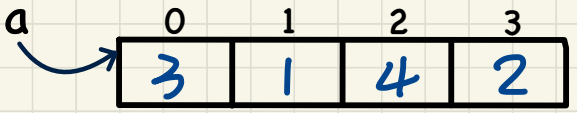
Inner Loop: select the next min from a[i] to a[n - 1] and put it to the end of the sorted region.

Outer Loop:
At the end of each iteration of the for-loop,
a is sorted from a[0] to a[i].

i	inner loop: j from ? to ?	midIndex at L6	after L6 - L8, a becomes?
			
			

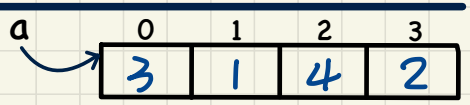
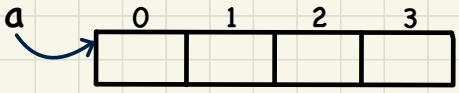
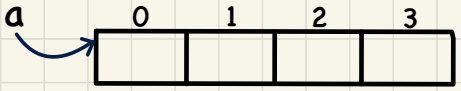
Insertion Sort in Java

```
1  void insertionSort(int[] a, int n)
2    for (int i = 1; i < n; i ++)
3      int current = a[i];
4      int j = i;
5      while (j > 0 && a[j - 1] > current)
6        a[j] = a[j - 1];
7        j --;
8      a[j] = current;
```

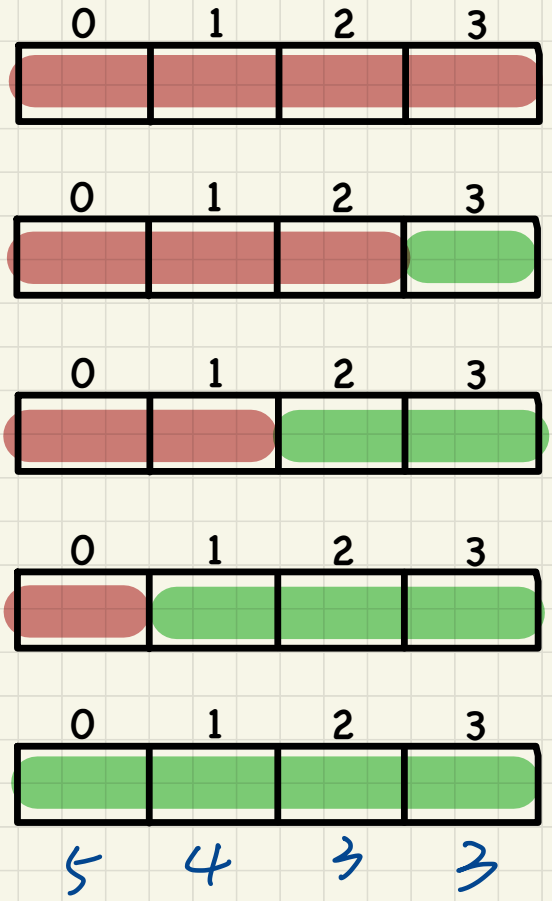


Inner Loop: find out where to insert current into a[0] to a[i] s.t. that part of a becomes sorted.

Outer Loop:
At the end of each iteration of the for-loop,
a is sorted from a[0] to a[i].

i	current after L3	j at L8	after L8, a becomes?
			
			
			

Exercise: Selection Sort vs. Insertion Sort



(1) sorted portion on the R

(2) non-ascending

Singly-Linked Lists (SLL): Visual Introduction

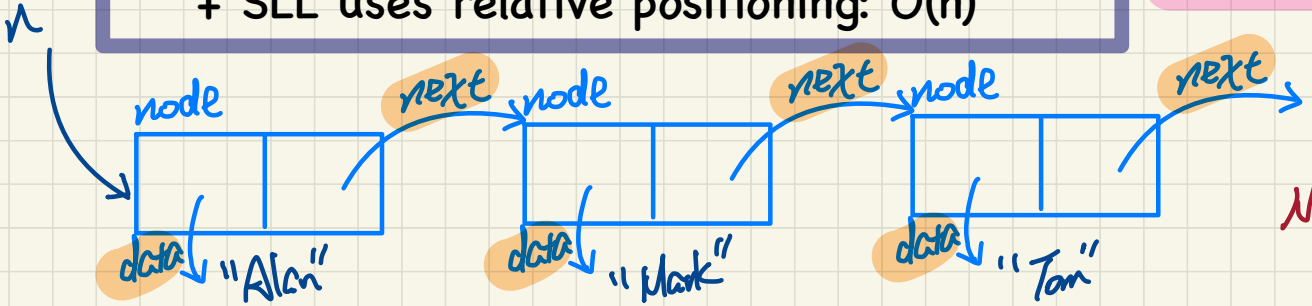
- A chain of connected nodes (via aliasing)
- Each node contains:
  - + reference to a data object
  - + reference to the next node
- Head vs. Tail
- The chain may grow or shrink dynamically.
- Accessing a position in a linear collection:
  - + Array uses absolute indexing: O(1)
  - + SLL uses relative positioning: O(n)

n: 1st node
n.data == "Alan"

n.next != null
n.next.data == "Mark"

n.next.next != null
n.next.next.data == "Tom"

n.next.next.next.data
↓
NullPointerException



Lecture 9 - February 4

Arrays and Linked Lists

Q: Mixing Insertion & Selection Sorts
SLL: Visual Introduction & Operations
SLL in Java: Node vs. SinglyLinkedList

Announcements/Reminders

- **Assignment 2** (on **SLL**) to be released soon
- **Assignment 1** solution released
- ***splitArrayHarder***: an **extended** version released
- Lecture notes template available
- Office Hours: 3pm to 4pm, Mon/Tue/Wed/Thu
- Contact Information of TAs on common eClass site

Exercise: Mixing the "Best" from both Sorts?

(selections)

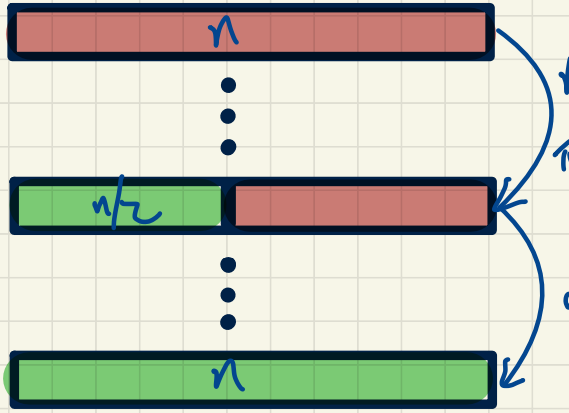
Recall:

- In insertion sort, costs of insertions are increasing.
- In selection sort, costs of selections are decreasing.

Idea:

- Perform insertion sort until half of the input is sorted.
- Perform selection sort to finish sorting the remaining half.

Q: Will this "new" algorithm perform better than $O(n^2)$?



$O(n^2 + n^2) = O(n^2)$

$O(\underbrace{1 + 2 + \cdots + n/2}_{\frac{(1+n/2)\cdot n/2}{2}}) = O(n^2)$

1st insertion, $n/2$th insertion

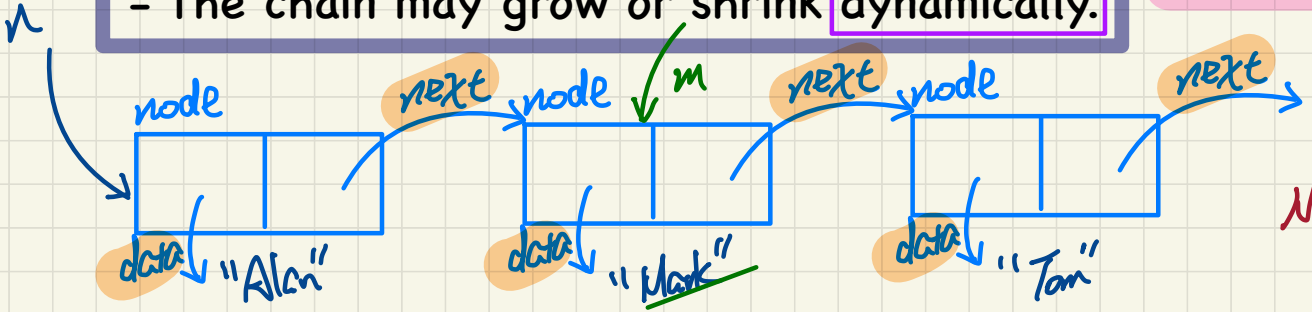
$O(\underbrace{n/2 + (n/2 - 1) + \cdots + 1}_{((n/2+1)\cdot n/2)/2}) = O(n^2)$

Singly-Linked Lists (SLL): Visual Introduction

- A chain of connected nodes (via aliasing)
- Each node contains:
  - + reference to a data object
  - + reference to the next node
- Head vs. Tail
- Accessing a position in a linear collection:
  - + Array uses absolute indexing: O(1)
  - + SLL uses relative positioning: O(n)
- The chain may grow or shrink dynamically.

e.g. m = n.next

n: 1st node
n.data == "Alan"
n.next != null
n.next.data == "Mark"
n.next.next != null
n.next.next.data == "Tom"
n.next.next.next.data
→ NullPointerException



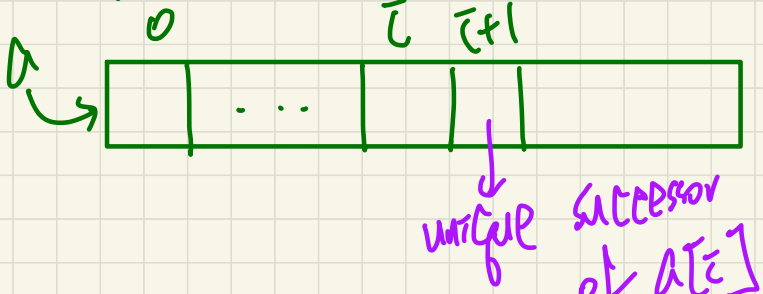
Linear vs. Non-linear collections

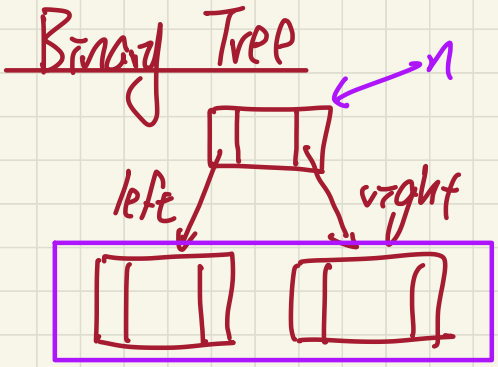
Linear

each position in the collection has a unique successor.

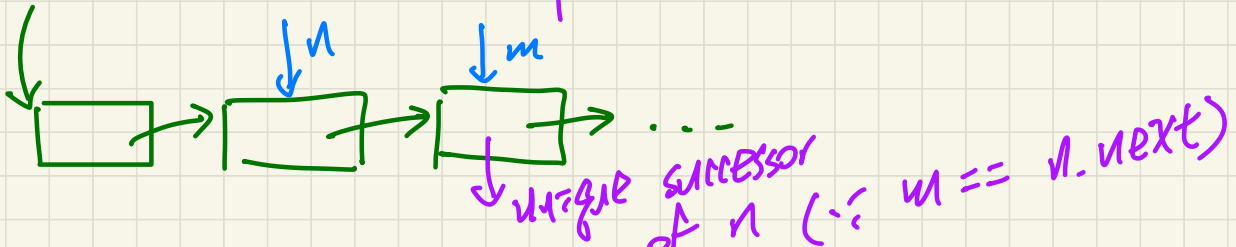
Non-linear

each position has multiple successors





multiple successors of n



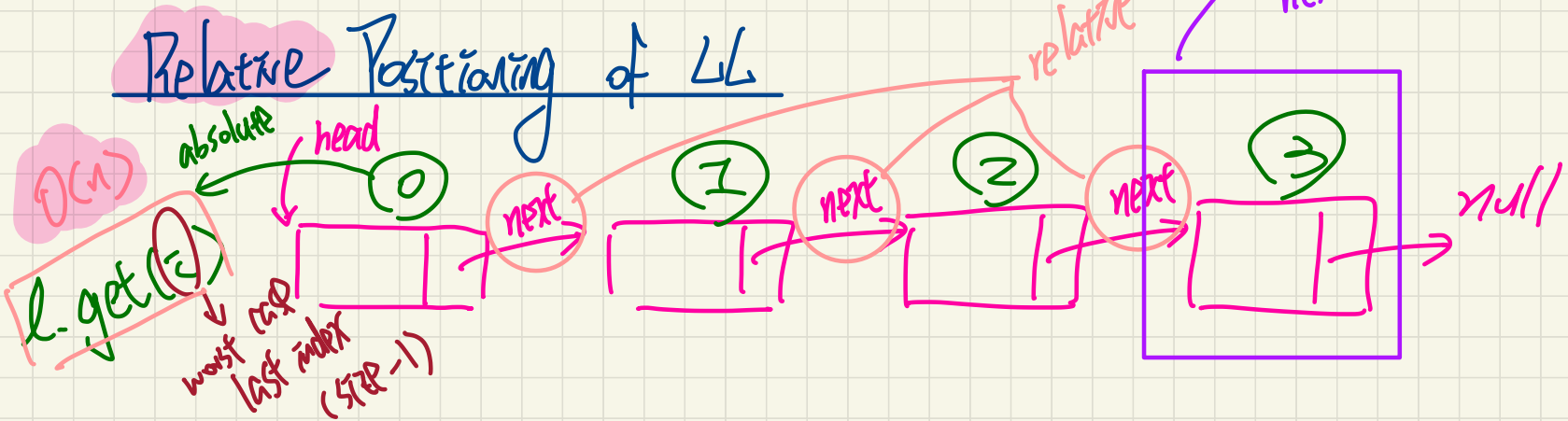
Absolute Indexing of Arrays

a[i] → O(1)

↓

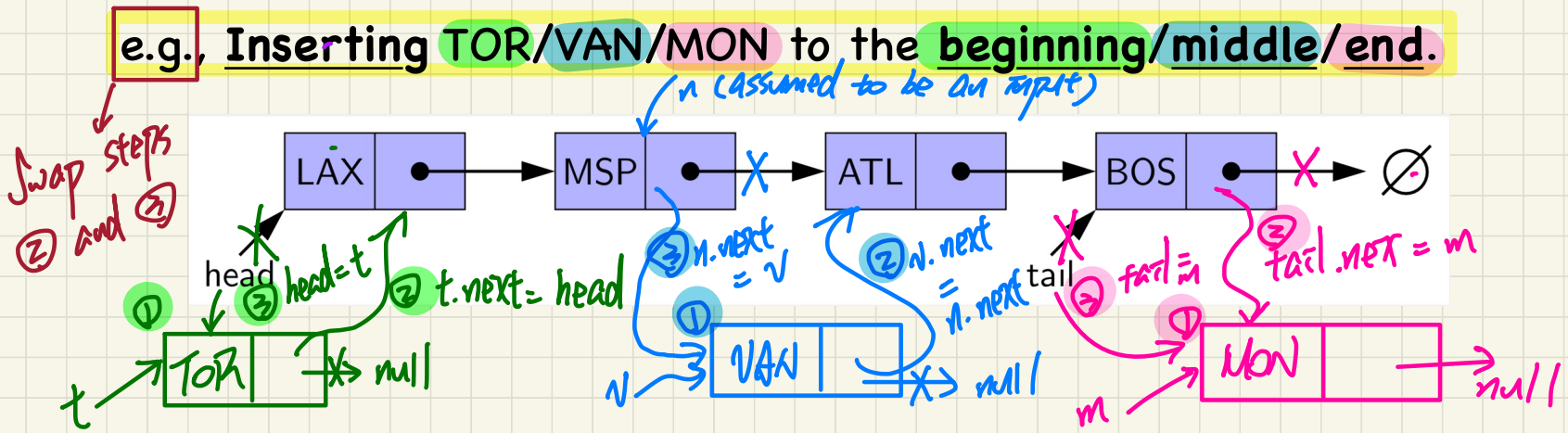
Tut: calculate the address offset from beginning of array.

Relative Positioning of LL

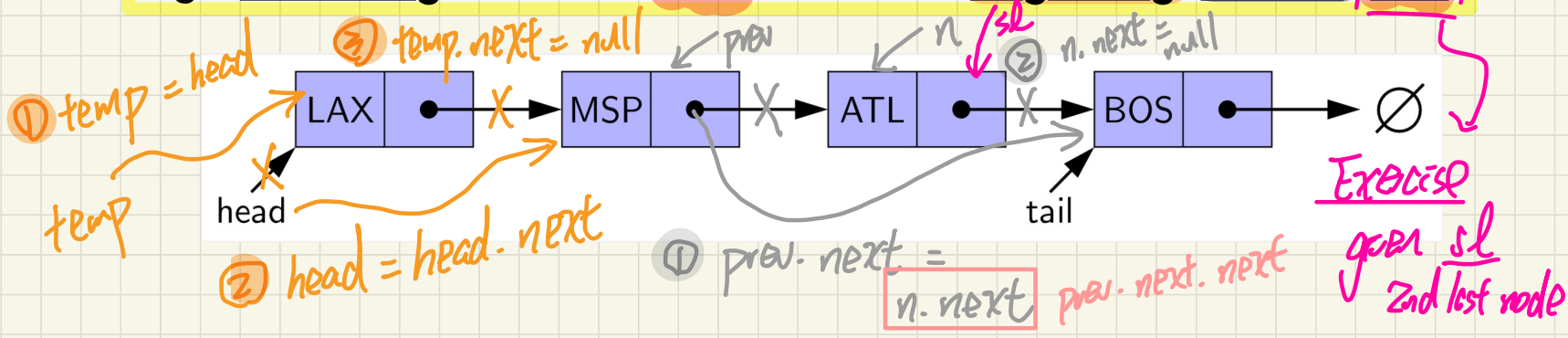


A SLL Grows or Shrinks Dynamically

e.g., Inserting TOR/VAN/MON to the beginning/middle/end.



e.g., Removing LAX/ATL/BOS from the beginning/middle/end.



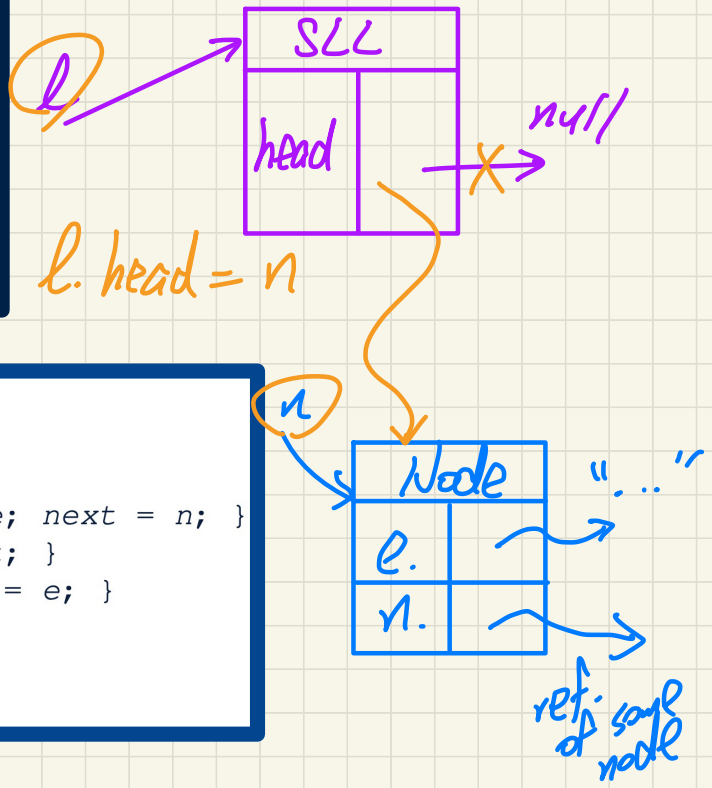
Implementing SLL in Java: SinglyLinkedList vs. Node

```
public class SinglyLinkedList {
  private Node head = null;
  public void setHead(Node n) { head = n; }
  public int getSize() { ... }
  public Node getTail() { ... }
  public void addFirst(String e) { ... }
  public Node getNodeAt(int i) { ... }
  public void addAt(int i, String e) { ... }
  public void removeLast() { ... }
}
```

recursive type

```
public class Node {
  private String element;
  private Node next;
  public Node(String e, Node n) { element = e; next = n; }
  public String getElement() { return element; }
  public void setElement(String e) { element = e; }
  public Node getNext() { return next; }
  public void setNext(Node n) { next = n; }
}
```

Runtime



SLL: Constructing a Chain of Nodes

Alan → Mark → Tom

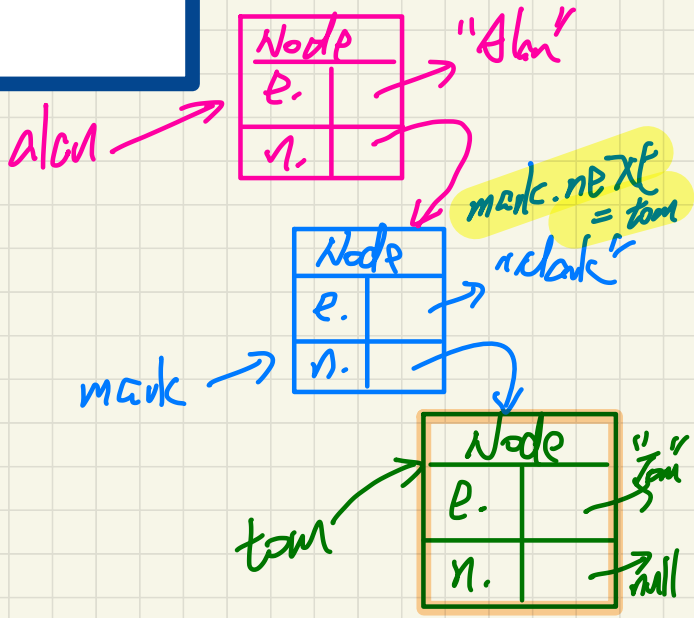
```
public class Node {
   private String element;
   private Node next;
   public Node(String e, Node n) { element = e; next = n; }
   public String getElement() { return element; }
   public void setElement(String e) { element = e; }
   public Node getNext() { return next; }
   public void setNext(Node n) { next = n; }
}
```

Aliasing

1. tom
2. mark.next
3. alan.next.next

Approach 1

```
Node tom = new Node("Tom", null);
Node mark = new Node("Mark", tom);
Node alan = new Node("Alan", mark);
```



Lecture 10 - February 6

Arrays and Linked Lists

SLL: List Constructions
SLL: getSize and getTail
Trading Space for Time: tail and size
SLL: addFirst

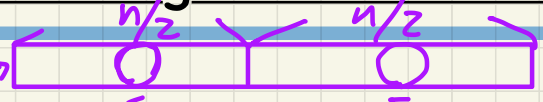
Announcements/Reminders

- **Assignment 2** (on **SLL**) released
  + **Required** studies: Generics in Java (Slides 33 – 36)
  + **Recommended** studies: extra SLL problems
- **Assignment 1** solution released
- ***splitArrayHarder***: an **extended** version released
- Lecture notes template available
- Office Hours: 3pm to 4pm, Mon/Tue/Wed/Thu
- Contact Information of TAs on common eClass site

* Precondition: $\forall i, j \cdot 0 \le i < \frac{n}{2} \wedge \frac{n}{2} \le j < n \Rightarrow input[i] \le input[j]$

Exercise: Mixing the "Best" from both Sorts?

Recall:



$input[i] \le input[j]$

- In insertion sort, costs of insertions are increasing.
- In selection sort, costs of selections are decreasing.

Idea:

- Perform insertion sort until half of the input is sorted.
- Perform selection sort to finish sorting the remaining half.

Q: Will this "new" algorithm perform better than $O(n^2)$?

Not working

<2, 1, 7, 3, 4, 0>

insertions

<1, 2, 7, 3, 4, 0>

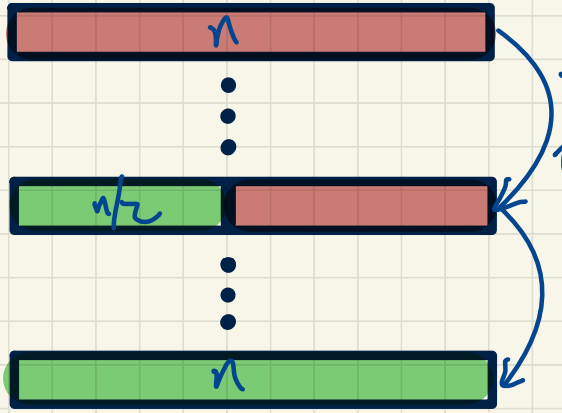
selections ✗

<1, 2, 7, 0, 3, 4>

Working

<2, 0, 1, 4, 7, 3>

$O(n^2 + n^2) = O(n^2)$



$O(\underbrace{1 + 2 + \cdots + n/2}_{\frac{(1 + n/2) \cdot n/2}{2}}) = O(n^2)$

1st insertion, $n/2$th insertion

$O(\underbrace{n/2 + (n/2 - 1) + \cdots + 1}_{((n/2 + 1) \cdot n/2)/2}) = O(n^2)$

SLL: Constructing a Chain of Nodes

Alan → Mark → Tom

```
public class Node {
   private String element;
   private Node next;
   public Node(String e, Node n) { element = e; next = n; }
   public String getElement() { return element; }
   public void setElement(String e) { element = e; }
   public Node getNext() { return next; }
   public void setNext(Node n) { next = n; }
}
```

Aliasing

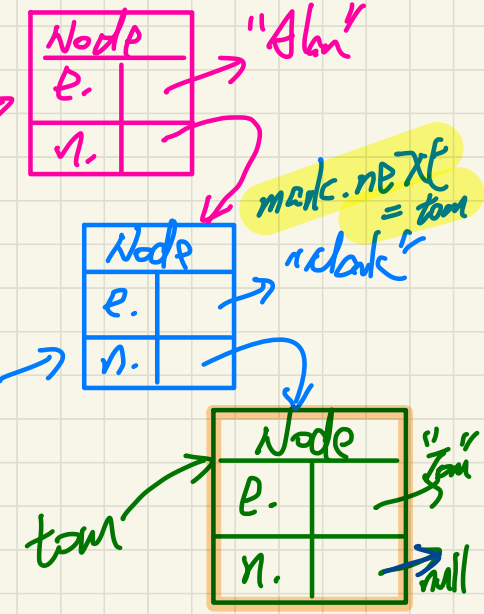
1. tom
2. mark.next
3. alan.next.next

Approach 1

```
Node tom = new Node("Tom", null);
Node mark = new Node("Mark", tom);
Node alan = new Node("Alan", mark);
```

unknown variable, compilation error

```
Node alan = new Node("Alan", mark);
Node mark = new Node("Mark", tom);
Node tom = new Node("Tom", null);
```

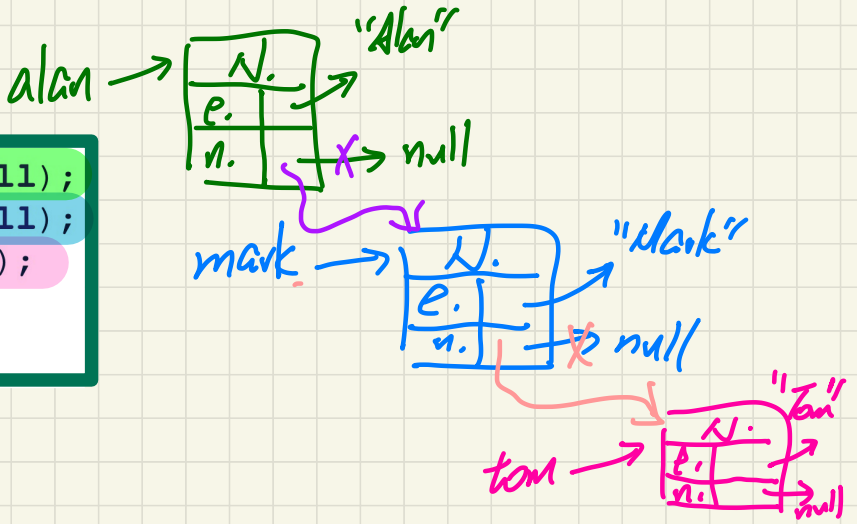


SLL: Constructing a Chain of Nodes

```
public class Node {
   private String element;
   private Node next;
   public Node(String e, Node n) { element = e; next = n; }
   public String getElement() { return element; }
   public void setElement(String e) { element = e; }
   public Node getNext() { return next; }
   public void setNext(Node n) { next = n; }
}
```

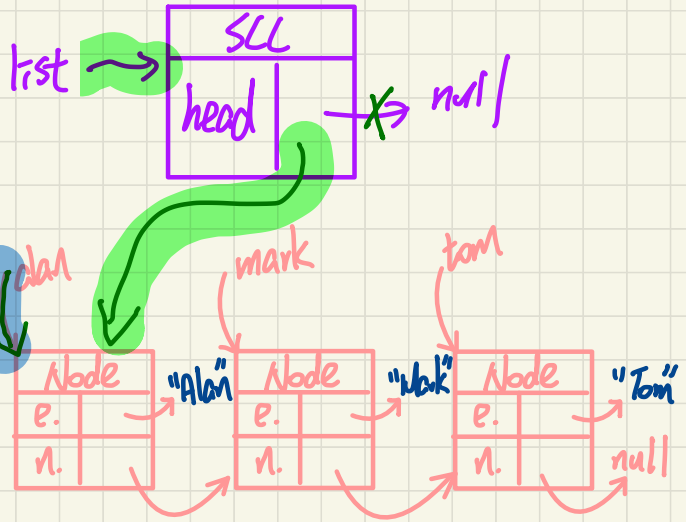
Approach 2

```
Node alan = new Node("Alan", null);
Node mark = new Node("Mark", null);
Node tom = new Node("Tom", null);
alan.setNext(mark);
mark.setNext(tom);
```



SLL: Setting a List's Head to a Chain of Nodes

```
public class SinglyLinkedList {
   private Node head = null;
   public void setHead(Node n) { head = n; }
   public int getSize() { ... }
   public Node getTail() { ... }
   public void addFirst(String e) { ... }
   public Node getNodeAt(int i) { ... }
   public void addAt(int i, String e) { ... }
   public void removeLast() { ... }
}
```



Approach 1

```
Node tom = new Node("Tom", null);
Node mark = new Node("Mark", tom);
Node alan = new Node("Alan", mark);
SinglyLinkedList list = new SinglyLinkedList();
list.setHead(alan);
```

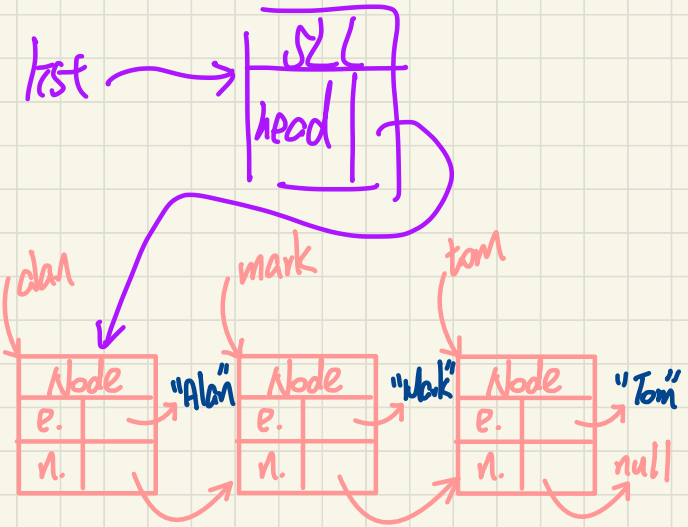
default const.

list.head == alan

T.

SLL: Setting a List's Head to a Chain of Nodes

```
public class SinglyLinkedList {
   private Node head = null;
   public void setHead(Node n) { head = n; }
   public int getSize() { ... }
   public Node getTail() { ... }
   public void addFirst(String e) { ... }
   public Node getNodeAt(int i) { ... }
   public void addAt(int i, String e) { ... }
   public void removeLast() { ... }
}
```

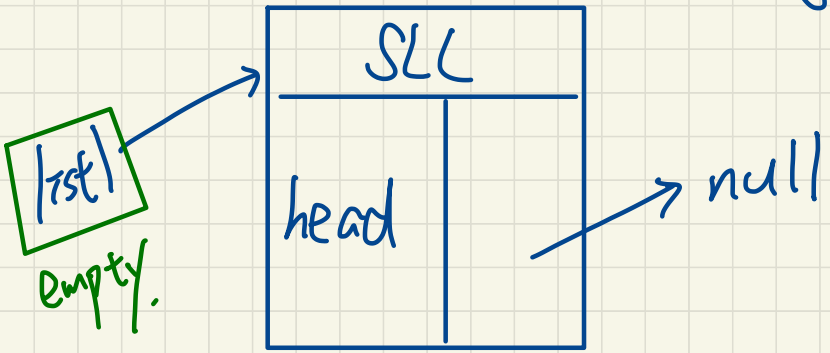


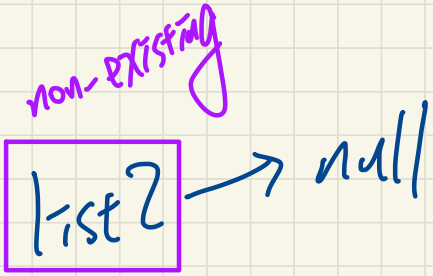
Approach 2

```
Node alan = new Node("Alan", null);
Node mark = new Node("Mark", null);
Node tom = new Node("Tom", null);
alan.setNext(mark);
mark.setNext(tom);
SinglyLinkedList list = new SinglyLinkedList();
list.setHead(alan);
```

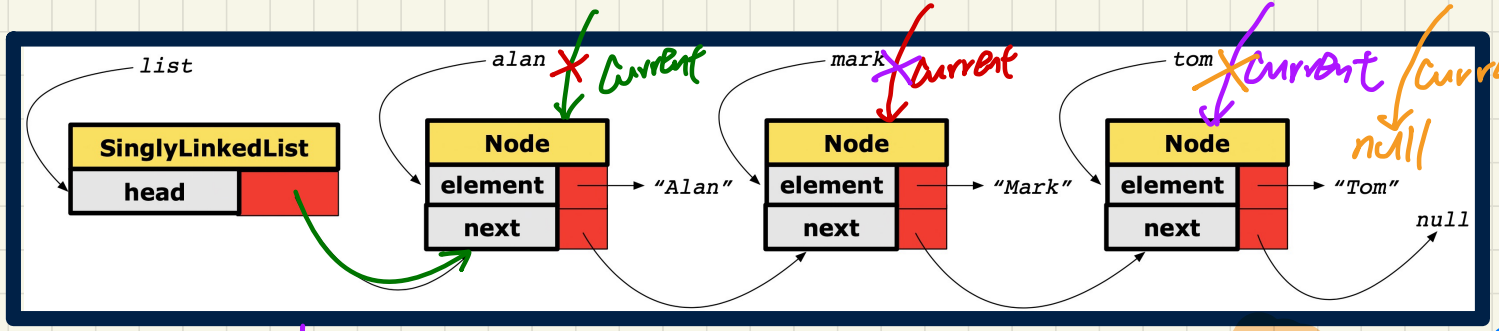
Empty SLL

```
SLL list1 = new SLL();
SLL list2 = null;
```





SLL Operation: Counting the Number of Nodes



a SLL method.

```
1  int getSize() {
2    int size = 0;
3    Node current = head;
4    while (current != null) {
5      current = current.getNext();
6      size ++;
7    }
8    return size;
9  }
```

≈ loop counter

exit when current == null

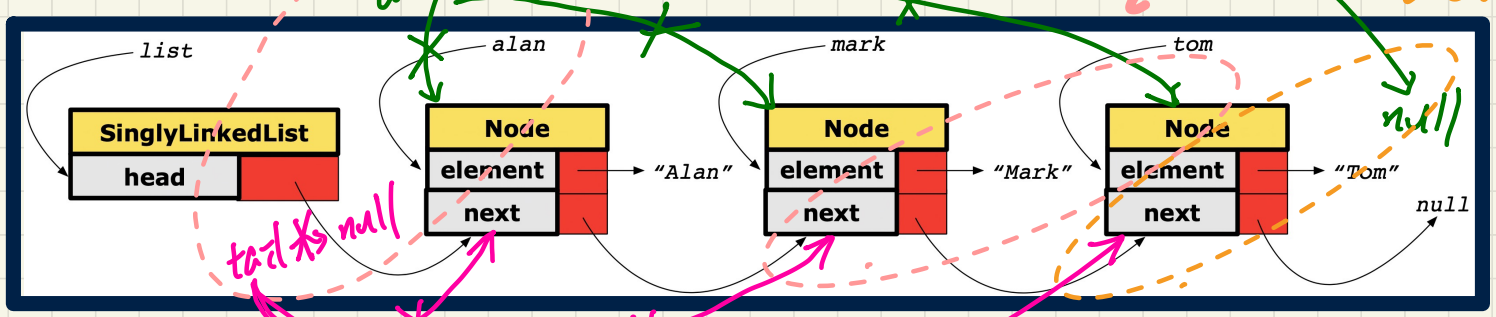
O(n) may be improved.

Trace: list.getSize()

O(n) # iterations = # nodes in chain

current	current != null	End of Iteration	size
alan	alan != null	current == mark	1
mark	mark != null	current == tom	2
tom	tom != null	current == null	3
null	null != null		
	F		

SLL Operation: Finding the Tail of the List




```
1  Node getTail() {
2      Node current = head;
3      Node tail = null;
4      while (current != null) {
5          tail = current;
6          current = current.getNext();
7      }
8      return tail;
9  }
```

Trace: list.getTail()

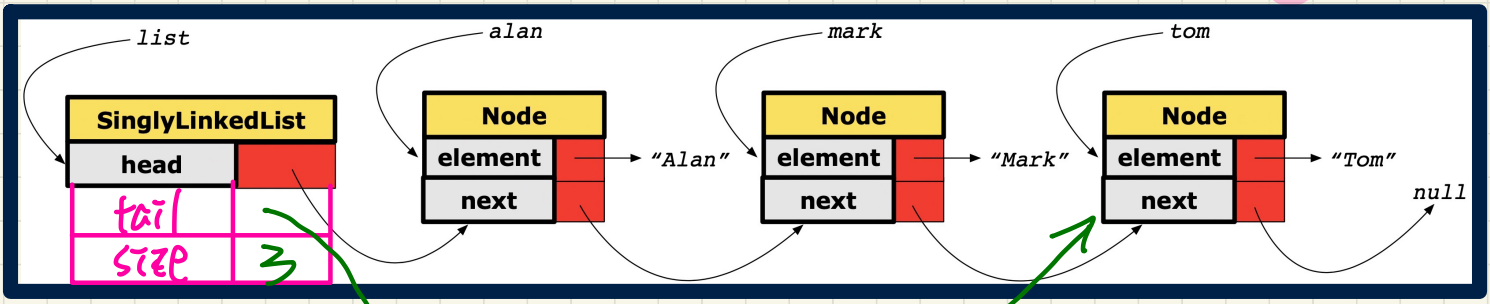
current	current != null	End of Iteration	tail

iterations = size of list

O(n)

SLL: Trading Space for Time

waste more memory space to make subsequent computations cheaper



Catch: For methods that might impact head, tail, or size of a SLL, the bodies of these methods must update them properly.

SLL class

attributes ⇓ more cost of space
- head
- tail
- size

SLL list = new SLL();

attribute access ⇓ less cost for running time
- list.tail
- list.size

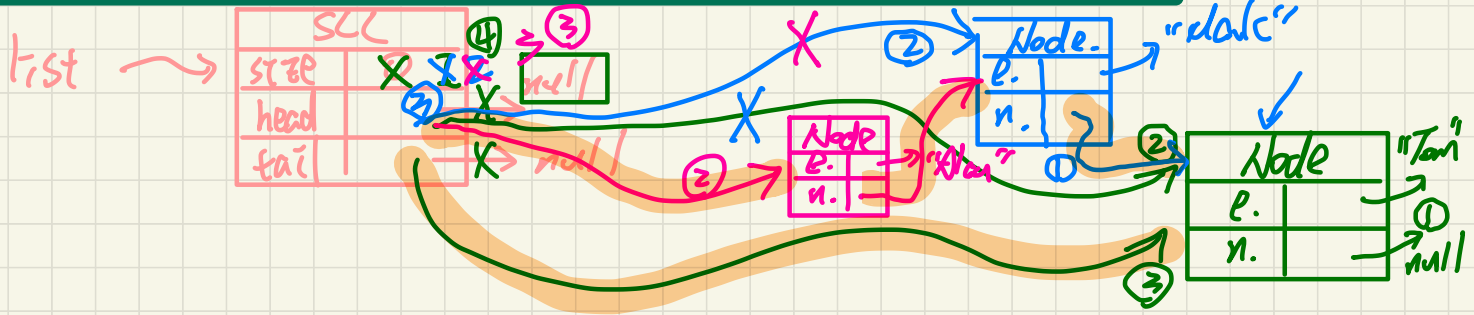
O(1)

Q. Does tail or size need to be updated?

SLL Operation: Inserting to the Front of the List

```java
@Test
public void testSLL_02() {
    SinglyLinkedList list = new SinglyLinkedList();
    assertTrue(list.getSize() == 0);
    assertTrue(list.getFirst() == null);

    list.addFirst("Tom");
    list.addFirst("Mark");
    list.addFirst("Alan");
    assertTrue(list.getSize() == 3);
    assertEquals("Alan", list.getFirst().getElement());
    assertEquals("Mark", list.getFirst().getNext().getElement());
    assertEquals("Tom", list.getFirst().getNext().getNext().getElement());
}
```

```java
1  void addFirst (String e) {
2    head = new Node(e, head);
3    if (size == 0) {
4      tail = head;
5    }
6    size ++;
7  }
```



Tutorial

Recursion Problem: splitArrayHarder

Coding in Java
Tracing
Exercises

Problem on Recursion

https://www.eecs.yorku.ca/~wangcw/teaching/lectures/2025/W/EECS2101/exercises/EECS2101-W25-Problem-Recursion-splitArray-Spec.pdf

A useful extension to the original `splitArray` problem:

+ Return an ArrayList of size 2:
+ If a split of equal sums (assumed to be unique) is possible:
  * index 0 of the returned list stores ArrayList of integers representing group 1.
  * index 1 of the returned list stores ArrayList of integers representing group 2.
+ If a split is not possible, both indices store empty lists.

e.g., **splitArrayHarder**({2, 2}) → <<2>, <2>>
e.g., **splitArrayHarder**({2, 3}) → <<>, <>>
e.g., **splitArrayHarder**({5, 2, 3}) → <<5>, <2, 3>>
e.g., **splitArrayHarder**({5, 2, 2}) → <<>, <>>

splitArrayHarder: Java Implementation

call by value

only modified in base case

```java
public ArrayList<ArrayList<Integer>> splitArrayHarder(int[] ns) {
    ArrayList<ArrayList<Integer>> output = new ArrayList<>();
    splitArrayHarderHelper(ns, 0, 0, 0, new ArrayList<Integer>(), new ArrayList<Integer>(), output);
    if(output.isEmpty()) {
        output.add(new ArrayList<Integer>());
        output.add(new ArrayList<Integer>());
    }
    return output;
}
```

g1 g2

either of them modified in every recursive call (non-base case)

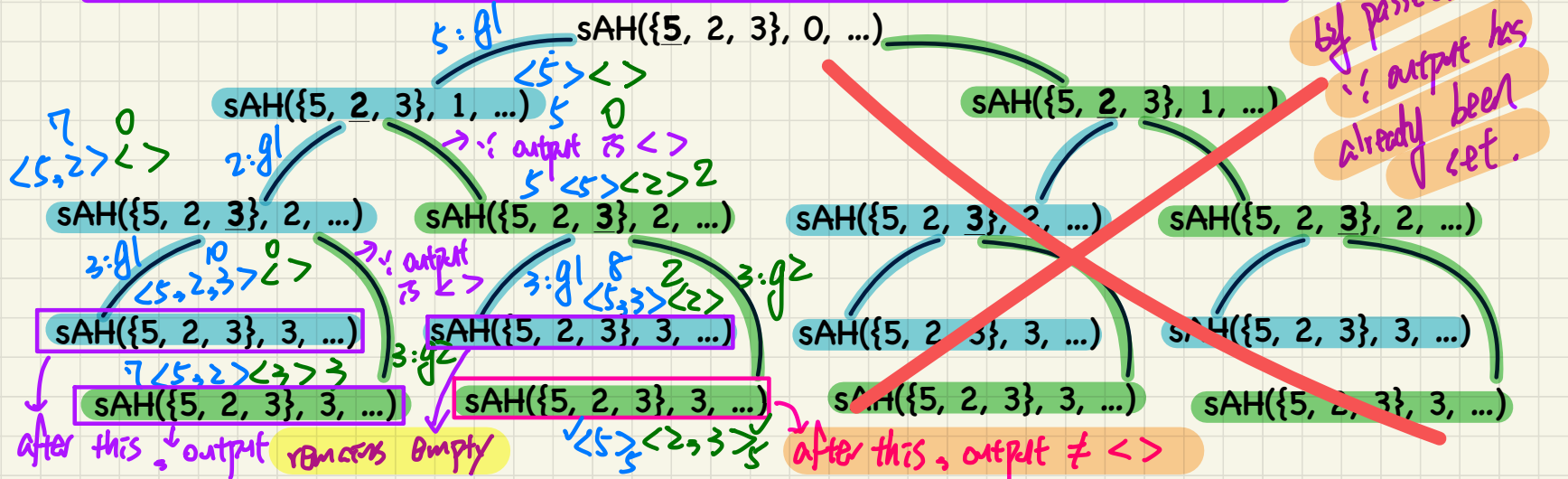
```java
private void splitArrayHarderHelper(int[] ns, int i, int sumOfGroup1, int sumOfGroup2,
        ArrayList<Integer> group1, ArrayList<Integer> group2, ArrayList<ArrayList<Integer>> output) {
    if(i == ns.length) {
        if(sumOfGroup1 == sumOfGroup2) {
            output.add(group1);
            output.add(group2);
        }
    }
    else {
        ArrayList<Integer> group1Extended = new ArrayList<>(group1);
        group1Extended.add(ns[i]);
        splitArrayHarderHelper(ns, i + 1, sumOfGroup1 + ns[i], sumOfGroup2, group1Extended, group2, output);

        if(output.isEmpty()) {
            ArrayList<Integer> group2Extended = new ArrayList<>(group2);
            group2Extended.add(ns[i]);
            splitArrayHarderHelper(ns, i + 1, sumOfGroup1, sumOfGroup2 + ns[i], group1, group2Extended, output);
        }
    }
}
```

splitArrayHarder: Tracing

Exercise: Trace in Eclipse

```
@Test
public void testSplitArrayHarder_03() {
    RecursiveMethods rm = new RecursiveMethods();
    int[] input = {5, 2, 3};
    ArrayList<ArrayList<Integer>> output = rm.splitArrayHarder(input);
    ArrayList<Integer> expectedGroup1 = new ArrayList<>(Arrays.asList(5));
    ArrayList<Integer> expectedGroup2 = new ArrayList<>(Arrays.asList(2, 3));
    assertTrue(output.size() == 2);
    assertEquals(expectedGroup1, output.get(0));
    assertEquals(expectedGroup2, output.get(1));
}
```

output: <<5>, <2,3>>



Lecture 11 - February 11

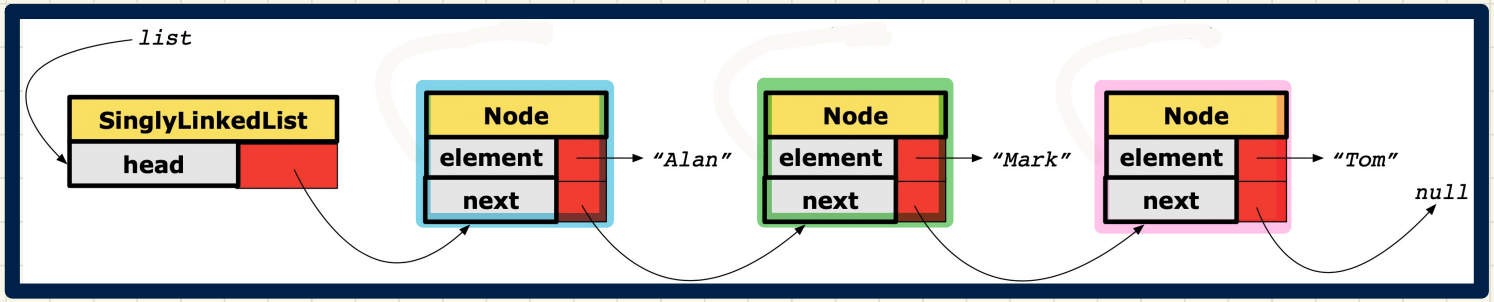
Arrays and Linked Lists

SLL: removeFirst, addLast
SLL: getNodeAt, insertAt, removeLast
Exercises: insertAfter, insertBefore

Announcements/Reminders

- ProgTest1 guide & example questions to be released
- *splitArrayHarder*: solution and tutorial video released
- Assignment 2 (on SLL) released
  + Required studies: Generics in Java (Slides 33 – 36)
  + Recommended studies: extra SLL problems
- Assignment 1 solution released
- Lecture notes template, Office Hours, TA Contact

SLL: Setting a List's Head to a Chain of Nodes



Approach 3

anonymous objects

Q. Given: SinglyLinkedList list = **new** SinglyLinkedList();
Write a **single line** of Java code to construct the above chain.

```
list.setHead(new Node("Alan", new Node("Mark", new Node("Tom", null))));
```

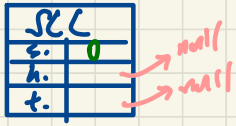
SLL Operation (sketch): Removing the First Node

void removeFirst()

Boundary Cases?

↳ size == 0 → Exception

↳ size == 1 → result: empty list



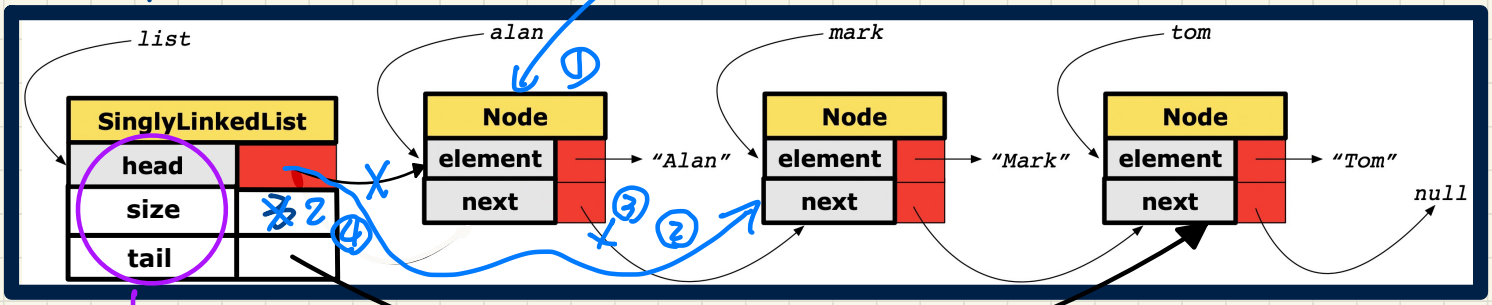
General Cases?

↳ size >= 2

① , ② , ③ , ④

∴ algo. does not depend on the size of the chain

O(1)

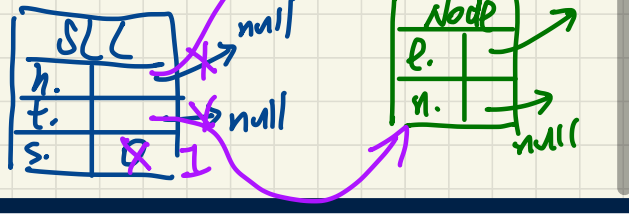


SLL Operation (sketch): Adding a Last Node

void addLast(String e)

Boundary Cases?

↳ size == 0

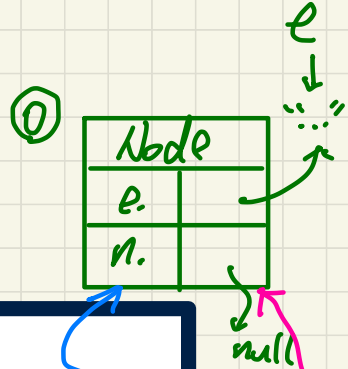


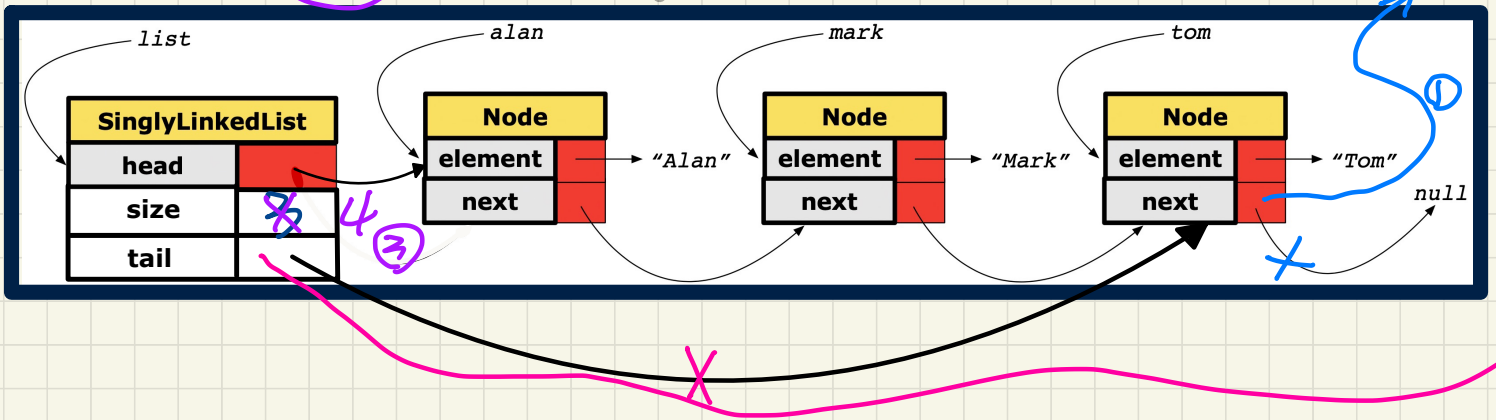
General Cases?

↳ size >= 1

O(1)

e ↓ ... @, ?, ≥, ≥.

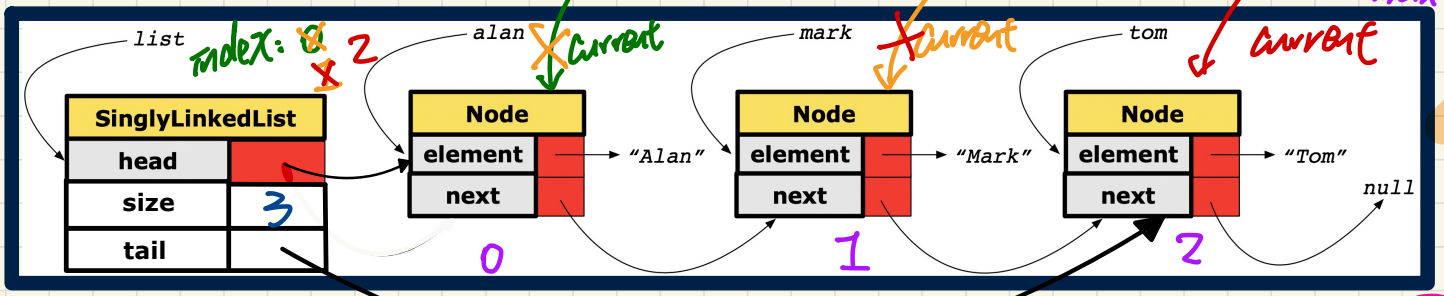




* when i = n-1 ⇒ list.tail . O(1) ; when i = n-2 ⇒ O(n-2) = O(n)

O((n-1) · 1) → O(n)

SLL Operation: Accessing the Middle of the List



0 ≤ i ≤ list.size-1

```
1  Node getNodeAt (int i) {
2    if (i < 0 || i >= size) { /* error
3    else {
4      int index = 0;
5      Node current = head;
6      while (index < i) { /* exit when
7        index ++;
8        current = current.getNext();
9      }
10     return current;
11   }
12 }
```

exit when index == i

Trace: list.getNodeAt(2)

current	index	index < 2	Start of Iteration
alan	0	0 < 2	1st it.: index 0 → 1, current alan →
mark	1	1 < 2	2nd it.: index 1 → 2, current mark → tom
tom	2	2 < 2	

exit.

① # it. = i

Q. Does **tail** or **size** need to be updated? No!

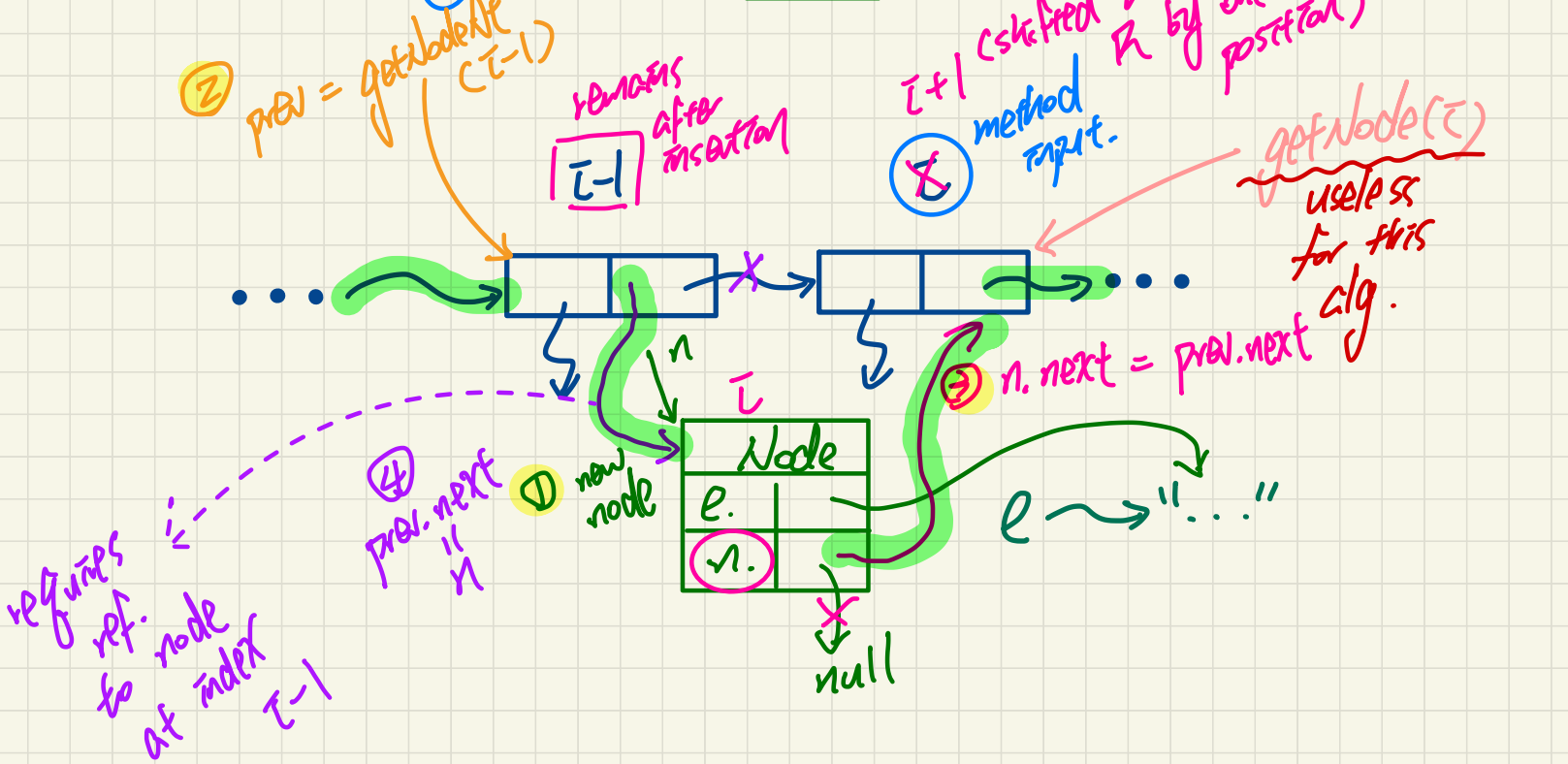
* ref to node at index i-1 not given

Idea of Inserting a Node at index i

* only an index is given

case i = 0 considered separately (i.e., addFirst(e))

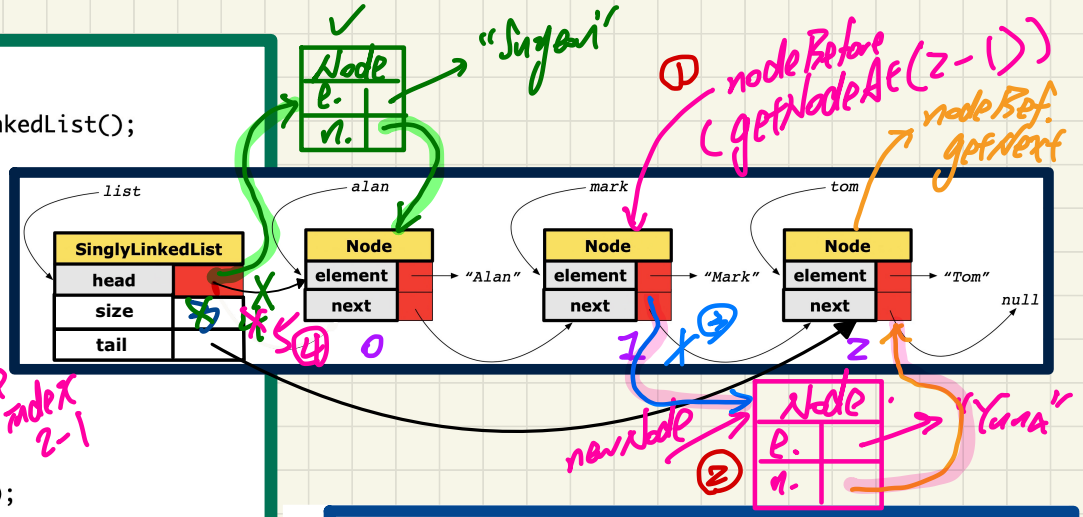
Case: addAt(i, e), where 0 < i ≤ size



SLL Operation: Inserting to the Middle of the List

```java
@Test
public void testSLL_addAt() {
    SinglyLinkedList list = new SinglyLinkedList();
    assertTrue(list.getSize() == 0);
    assertTrue(list.getFirst() == null);

    list.addFirst("Tom");
    list.addFirst("Mark");
    list.addFirst("Alan");
    assertTrue(list.getSize() == 3);

    list.addAt(0, "Suyeon");
    list.addAt(2, "Yuna");
    assertTrue(list.getSize() == 5);
    list.addAt(list.getSize(), "Heeyeon");
    assertTrue(list.getSize() == 6);
    assertEquals("Suyeon", list.getNodeAt(0).getElement());
    assertEquals("Alan", list.getNodeAt(1).getElement());
    assertEquals("Yuna", list.getNodeAt(2).getElement());
    assertEquals("Mark", list.getNodeAt(3).getElement());
    assertEquals("Tom", list.getNodeAt(4).getElement());
    assertEquals("Heeyeon", list.getNodeAt(5).getElement());
}
```

need ref. to node at index 2-1

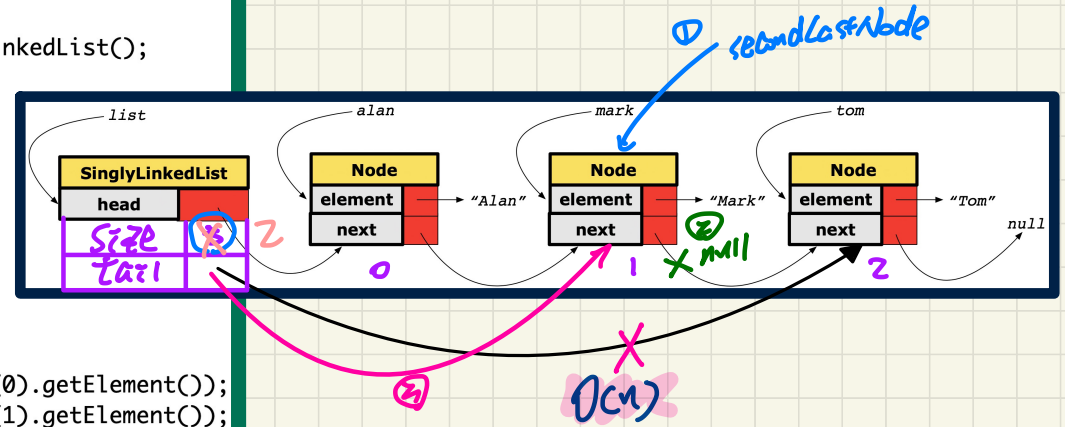



```java
1  void addAt (int i, String e) {
2    if (i < 0 || i > size) {
3      throw new IllegalArgumentException("Invalid Index.");
4    }
5    else {
6      if (i == 0) {
7        addFirst(e);
8      }
9      else {
10       Node nodeBefore = getNodeAt(i - 1);
11       Node newNode = new Node(e, nodeBefore.getNext());
12       nodeBefore.setNext(newNode);
13       size ++;
14     }
15   }
16 }
```

Q. Does **tail** or **size** need to be updated?

SLL Operation: Removing the End of the List

```java
@Test
public void testSLL_removeLast() {
    SinglyLinkedList list = new SinglyLinkedList();
    assertTrue(list.getSize() == 0);
    assertTrue(list.getFirst() == null);

    list.addFirst("Tom");
    list.addFirst("Mark");
    list.addFirst("Alan");
    assertTrue(list.getSize() == 3);

    list.removeLast();
    assertTrue(list.getSize() == 2);
    assertEquals("Alan", list.getNodeAt(0).getElement());
    assertEquals("Mark", list.getNodeAt(1).getElement());

    list.removeLast();
    assertTrue(list.getSize() == 1);
    assertEquals("Alan", list.getNodeAt(0).getElement());

    list.removeLast();
    assertTrue(list.getSize() == 0);
    assertNull(list.getFirst());
}
```




```java
1  void removeLast () {
2    if (size == 0) {
3    throw new IllegalArgumentException("Empty List.");
4    }
5    else if (size == 1) {
6    removeFirst();
7    }
8    else {
9    Node secondLastNode = getNodeAt(size - 2);
10   secondLastNode.setNext(null);
11   tail = secondLastNode;
12   size --;
13   }
14 }
```

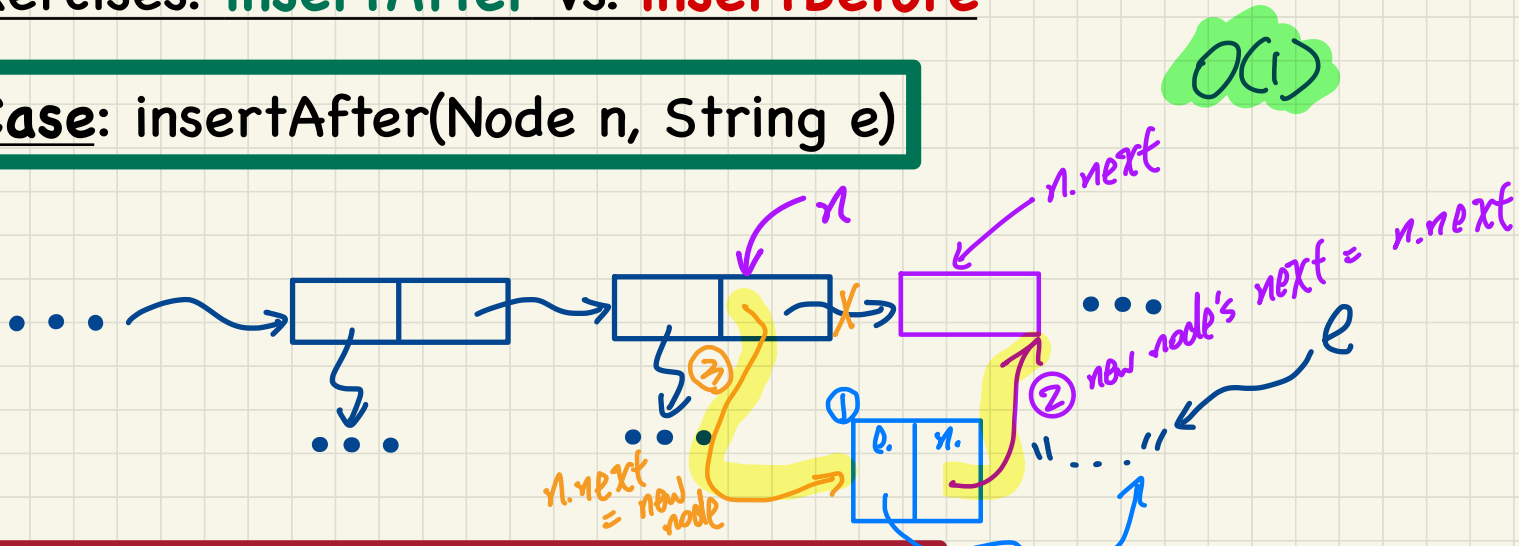
O(n-2) = O(n)

n-2=1

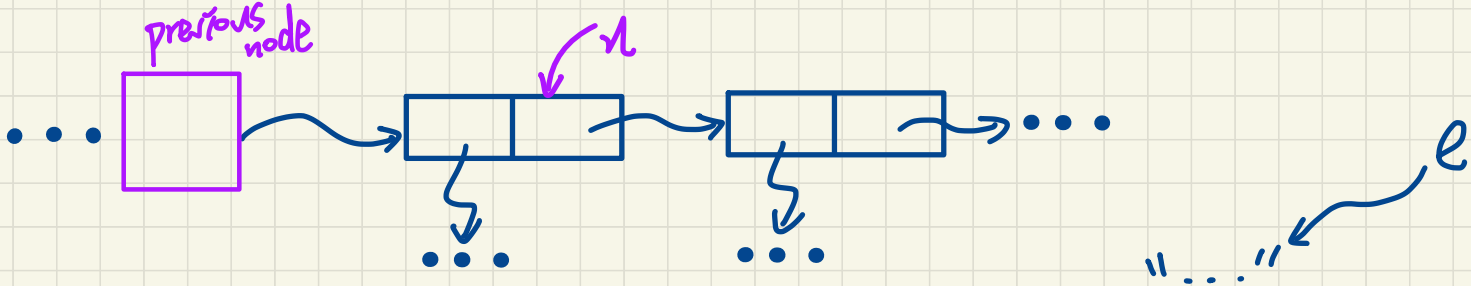
Q. Does tail or size need to be updated?

Exercises: insertAfter vs. insertBefore

Case: insertAfter(Node n, String e)



Case: insertBefore(Node n, String e)



Lecture 12 - February 13

Arrays and Linked Lists

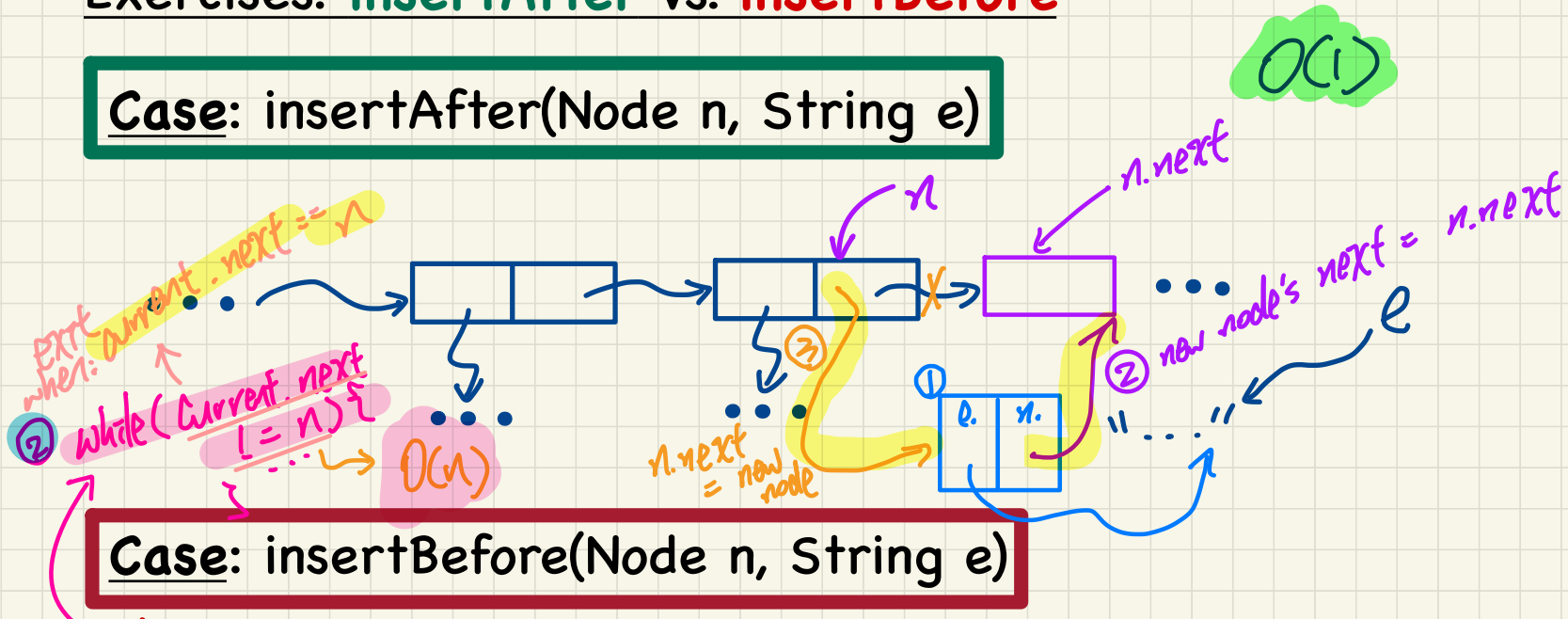
DLL: Introduction
DLL in Java: Node vs. Doubly-Linked Lists
DLL in Java: addBetween, remove

Announcements/Reminders

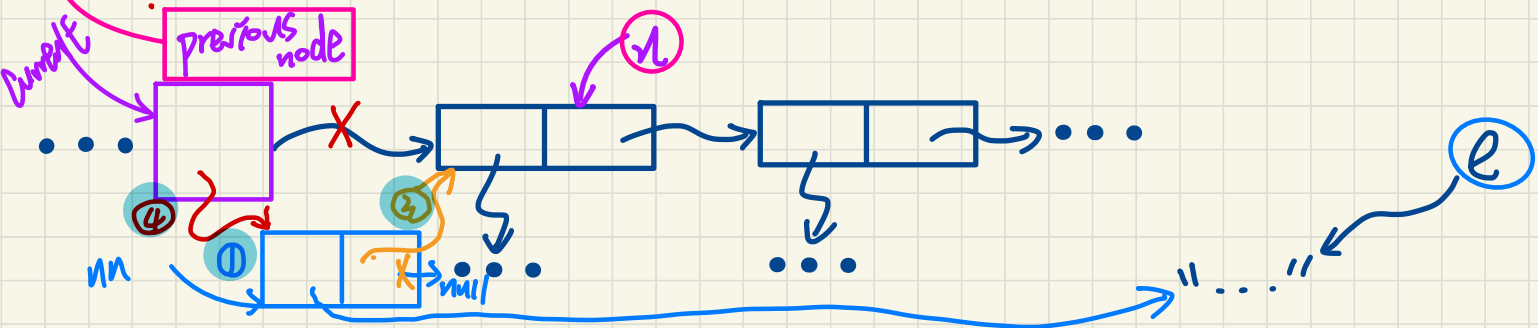
- **ProgTest1** guide & example questions released
- **In-person** Office Hours during RW to be announced
- ***splitArrayHarder***: solution and tutorial video released
- **Assignment 2** (on **SLL**) released
  + **Required** studies: Generics in Java (Slides 33 – 36)
  + **Recommended** studies: extra SLL problems
- **Assignment 1** solution released
- Lecture notes template, TA Contact

Exercises: insertAfter vs. insertBefore

Case: insertAfter(Node n, String e)



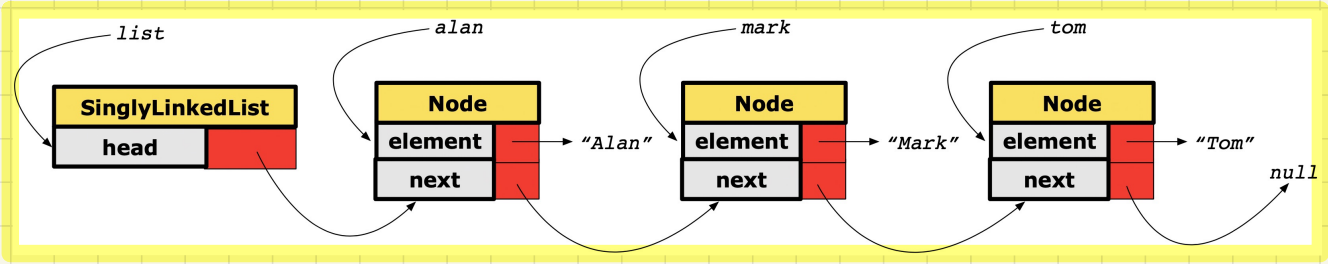
Case: insertBefore(Node n, String e)

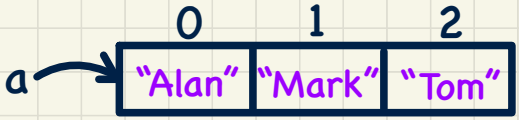


Running Time: Arrays vs. Singly-Linked Lists

Operation \ Data Structure		Array	Singly-Linked List
get size		O(1)	
get first/last element			
get element at index i		O(1)	O(n)
remove last element			
add/remove first element, add last element		O(n)	O(1)
add/remove i^{th} element	given reference to $(i-1)^{th}$ element		
	not given		O(n)

array: (no resizing) a.length-1 nulls

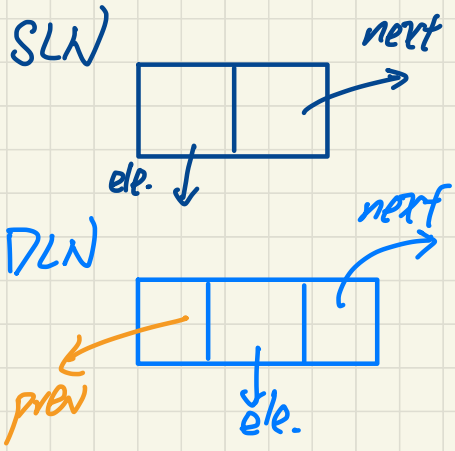




Doubly-Linked Lists (DLL): Visual Introduction

- A chain of bi-directionally connected nodes
- Each node contains:
  - + reference to a data object
  - + reference to the next node
  - + reference to the previous node
- A DLL is also a SLL:
  - + many methods implemented the same way
  - + some method implemented more efficiently
- Each DLL stores dedicated Header & Trailer Nodes (no head refeference and no tail reference)
- The chain may grow or shrink dynamically.
- Accessing a node in a DLL (via next or prev):
  - + Relative positioning: O(n)

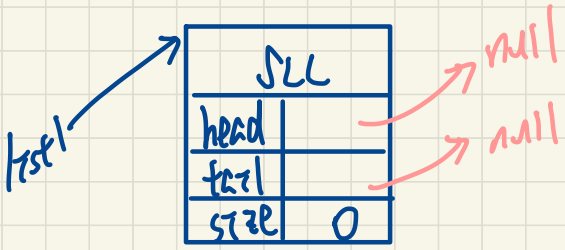
using next ref.



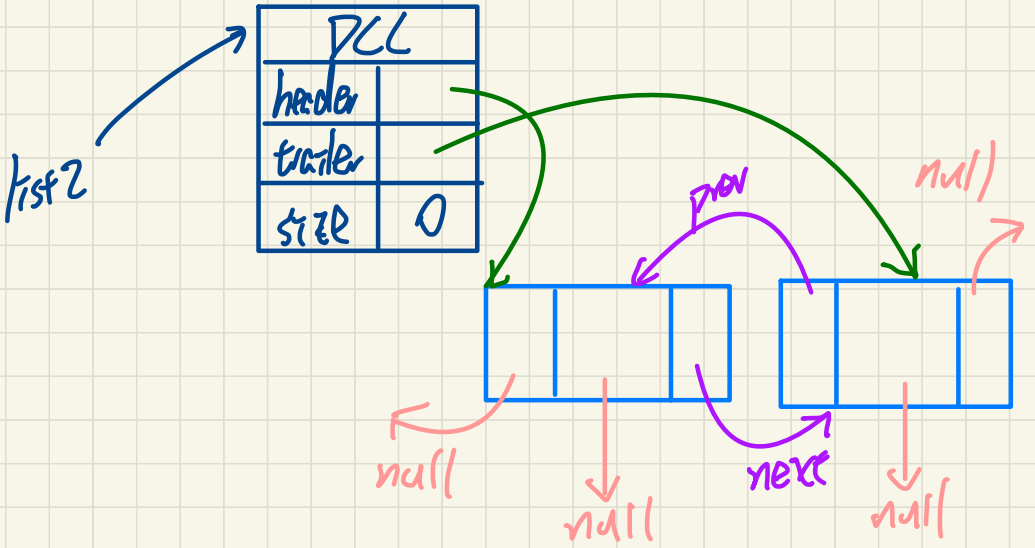
removeLast
(∵ 2nd last node → trailer.prev.prev)
O(1)

Empty Lists: SLLs vs. DLLs

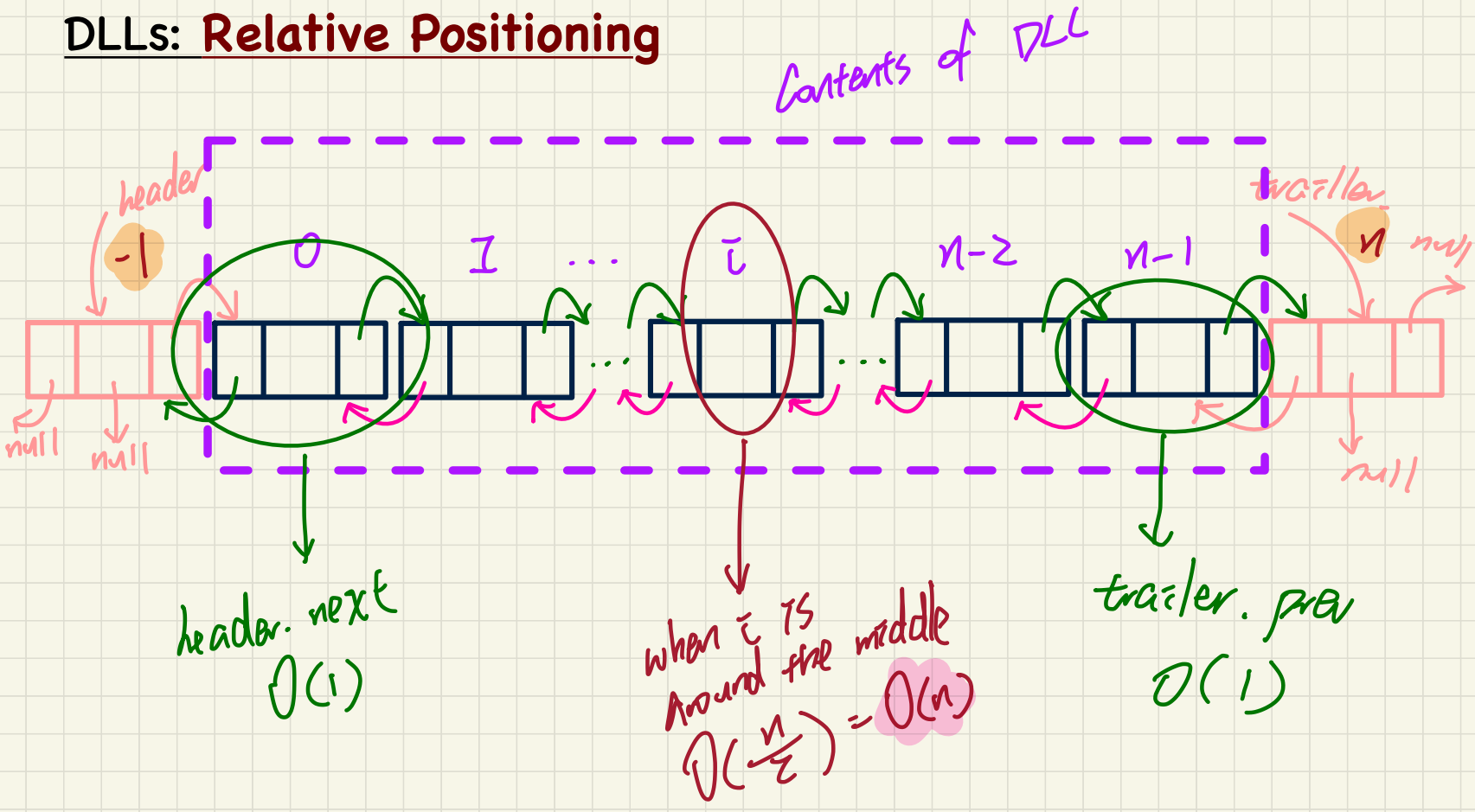
Empty SLL



Empty DLL



DLLs: Relative Positioning

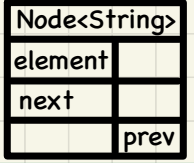


Why DLL/DLN

list node

1. performance (e.g. remove/csf)
   - ↳ prev ref. for DLN
2. code structure — don't need to handle edge cases explicitly.
   - ↳ header, trailer for DLL

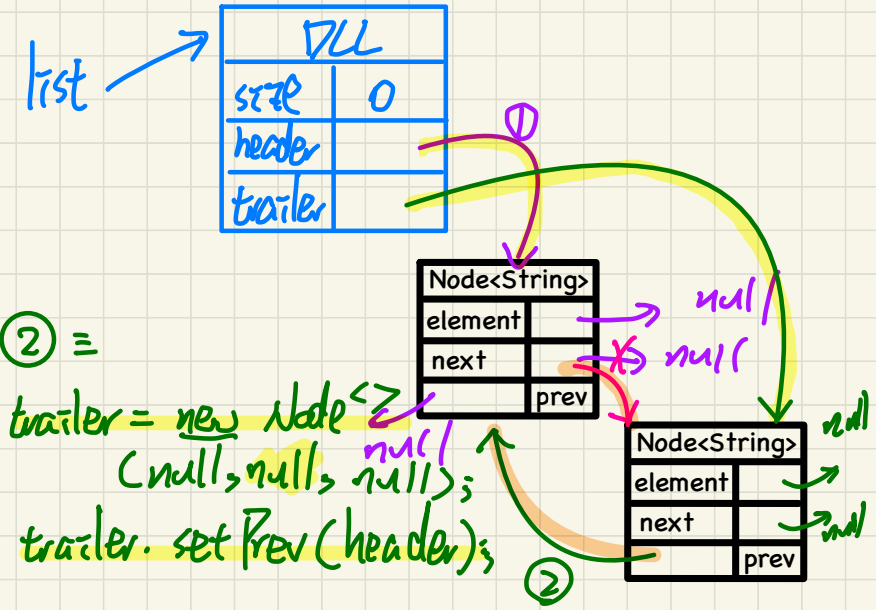
Generic DLL in Java: DoublyLinkedList vs. Node




```java
public class DoublyLinkedList<E> {
   private int size = 0;
   public void addFirst(E e) { ... }
   public void removeLast() { ... }
   public void addAt(int i, E e) { ... }
   private Node<E> header;
   private Node<E> trailer;
   public DoublyLinkedList() {
      header = new Node<>(null, null, null);
      trailer = new Node<>(null, header, null);
      header.setNext(trailer);
   }
}
```

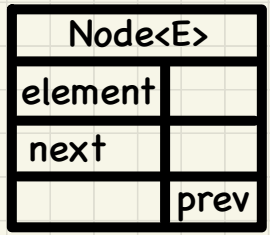
```java
public class Node<E> {
   private E element;
   private Node<E> next;
   public E getElement() { return element; }
   public void setElement(E e) { element = e; }
   public Node<E> getNext() { return next; }
   public void setNext(Node<E> n) { next = n; }
   private Node<E> prev;
   public Node<E> getPrev() { return prev; }
   public void setPrev(Node<E> p) { prev = p; }
   public Node(E e, Node<E> p, Node<E> n) {
      element = e;
      prev = p;
      next = n;
   }
}
```

```java
@Test
public void test_String_DLL_Empty_List() {
    DoublyLinkedList<String> list = new DoublyLinkedList<>();
    assertTrue(list.getSize() == 0);
    assertTrue(list.getFirst() == null);
    assertTrue(list.getLast() == null);
}
```



assumptions: pred.next == succ && succ.prev == pred

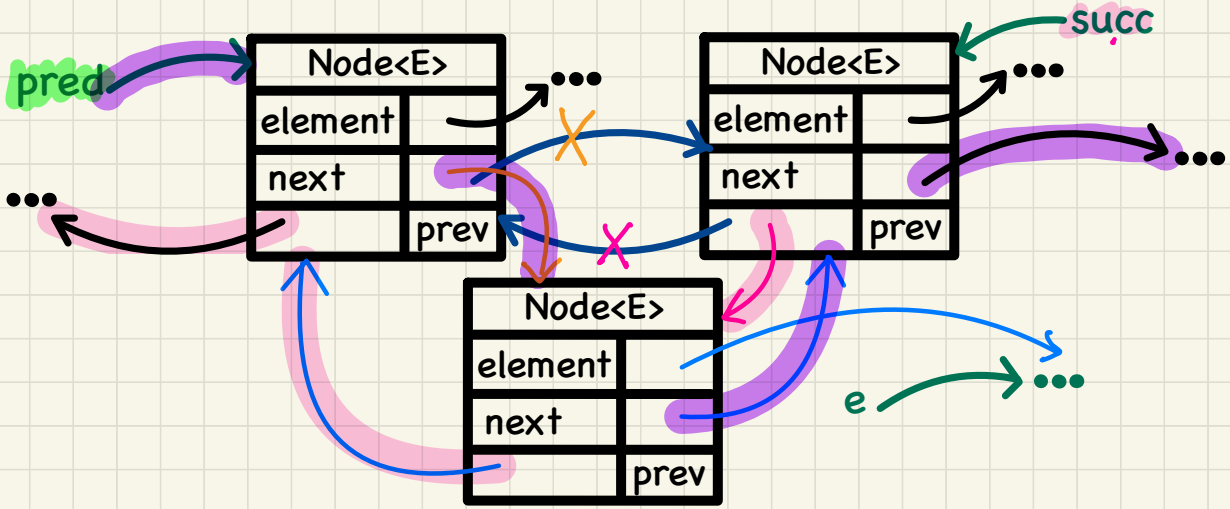
Generic DLL in Java: Inserting between Nodes



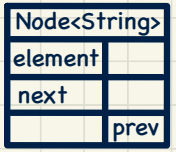

```
1  void addBetween(E e, Node<E> pred, Node<E> succ) {
2    Node<E> newNode = new Node<>(e, pred, succ);
3    pred.setNext(newNode);
4    succ.setPrev(newNode);
5    size ++;
6  }
```

prev. next

Assumption: pred and succ are directly connected.



Generic DLL in Java: Inserting to the Front/End




```java
@Test
public void test_String_DLL_Insert_Front_End() {
    DoublyLinkedList<String> list = new DoublyLinkedList<>();
    list.addFirst("Mark");
    list.addFirst("Alan");

    assertTrue(list.getSize() == 2);
    assertEquals("Alan", list.getFirst().getElement());
    assertEquals("Mark", list.getFirst().getNext().getElement());

    list = new DoublyLinkedList<>();
    list.addLast("Mark");
    list.addLast("Alan");

    assertTrue(list.getSize() == 2);
    assertEquals("Alan", list.getLast().getElement());
    assertEquals("Mark", list.getLast().getPrev().getElement());
}
```

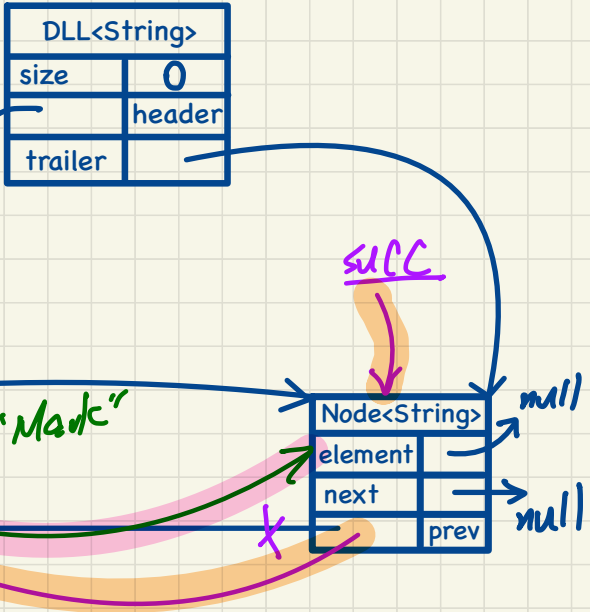
exercises

```java
void addFirst(E e) {
  addBetween(e, header, header.getNext())
}
```

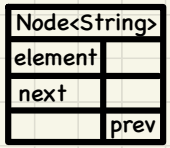
pred succ

```java
void addLast(E e) {
  addBetween(e, trailer.getPrev(), trailer)
}
```

pred succ.



Generic DLL in Java: Inserting to the Middle




```java
@Test
public void test_String_DLL_addAt() {
    DoublyLinkedList<String> list = new DoublyLinkedList<>();
    list.addAt(0, "Alan");
    list.addAt(1, "Tom");
    list.addAt(1, "Mark");

    assertTrue(list.getSize() == 3);
    assertEquals("Alan", list.getFirst().getElement());
    assertEquals("Mark", list.getFirst().getNext().getElement());
    assertEquals("Tom", list.getFirst().getNext().getNext().getElement());
}
```

tracing exercise.

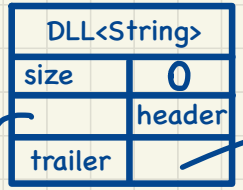
```java
addAt(int i, E e) {
  if (i < 0 || i > size) {
    throw new IllegalArgumentException
  else {
    Node<E> pred = getNodeAt(i - 1);
    Node<E> succ = pred.getNext();
    addBetween(e, pred, succ);
  }
}
```

pred at index i-1

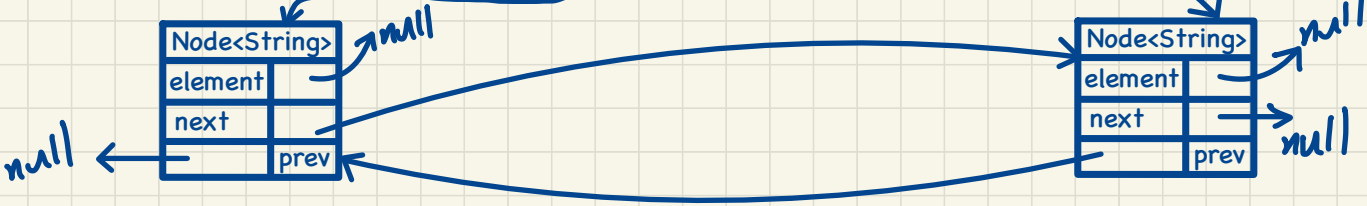
succ. at index i

Notes.

+ getNodeAt(-1) returns the **header**
+ getNodeAt(size) returns the **trailer**



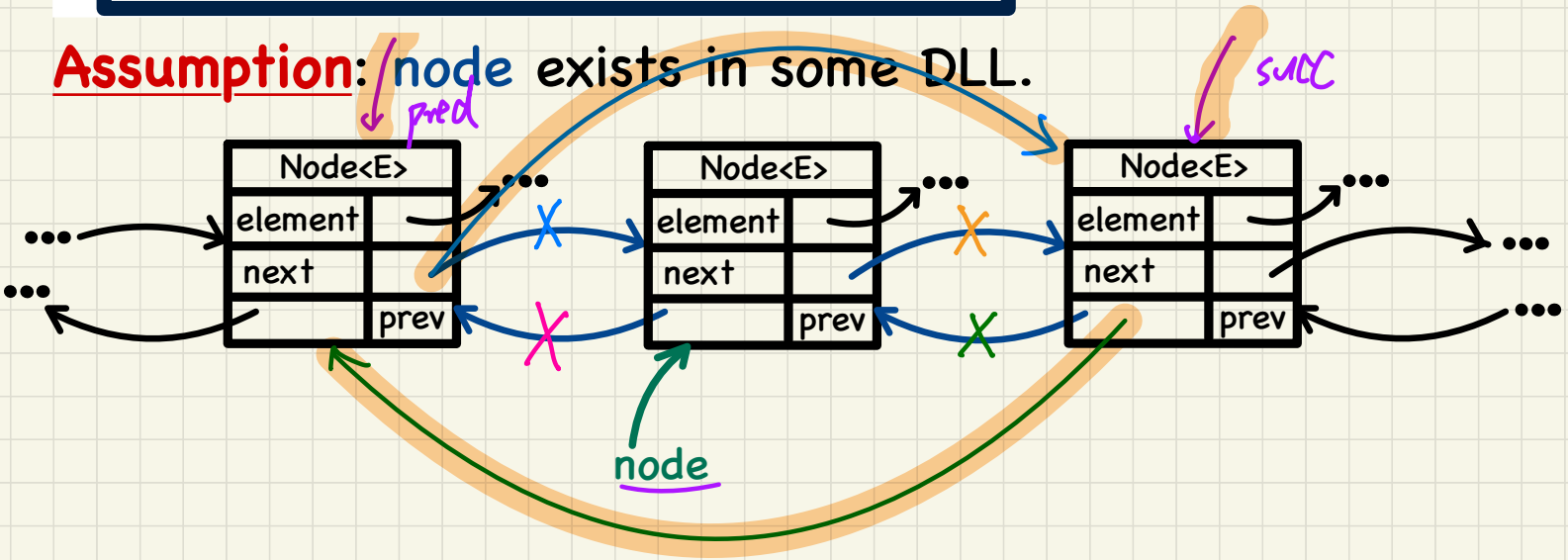
worst case: i-1 ≈ n/2 → O(n)



Generic DLL in Java: Removing a Node

```
1  void remove(Node<E> node) {
2    Node<E> pred = node.getPrev();
3    Node<E> succ = node.getNext();
4    pred.setNext(succ);
5    succ.setPrev(pred);
6    node.setNext(null);
7    node.setPrev(null);
8    size --;
9  }
```

Assumption: node exists in some DLL.



Generic DLL in Java: Removing from the Front/End

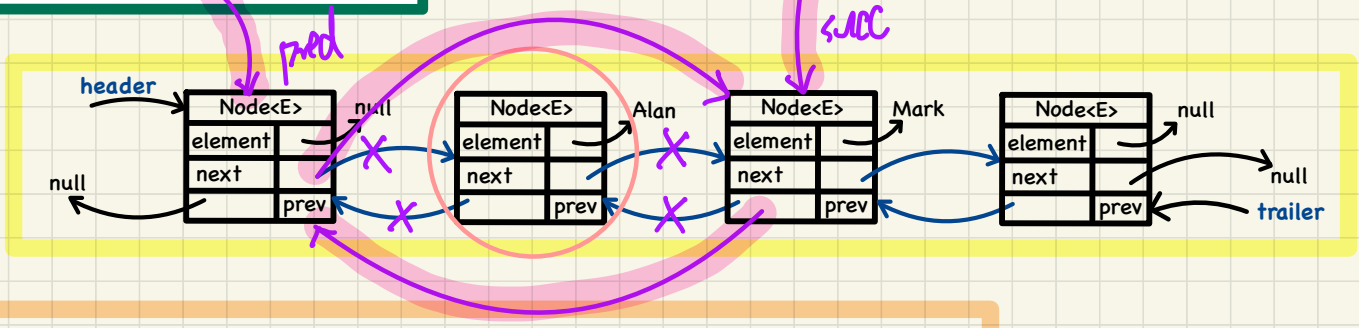
```java
@Test
public void test_String_DLL_Remove_Front_End() {
    DoublyLinkedList<String> list = new DoublyLinkedList<>();
    list.addFirst("Mark");
    list.addFirst("Alan");
    list.removeFirst();
    list.removeFirst();
    assertTrue(list.getSize() == 0);

    list = new DoublyLinkedList<>();
    list.addFirst("Mark");
    list.addFirst("Alan");
    list.removeLast();
    list.removeLast();
    assertTrue(list.getSize() == 0);
}
```

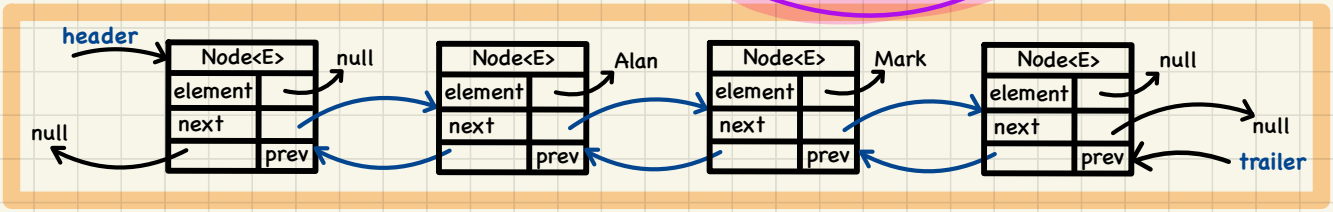
tracing / execute

1st node

```java
void removeFirst() {
  if (size == 0) { throw new IllegalArgumentException("Empty"); }
  else { remove(header.getNext()); }
}
```

```java
void removeLast() {
  if (size == 0) { throw new IllegalArgumentException("Empty"); }
  else { remove(trailer.getPrev()); }
}
```



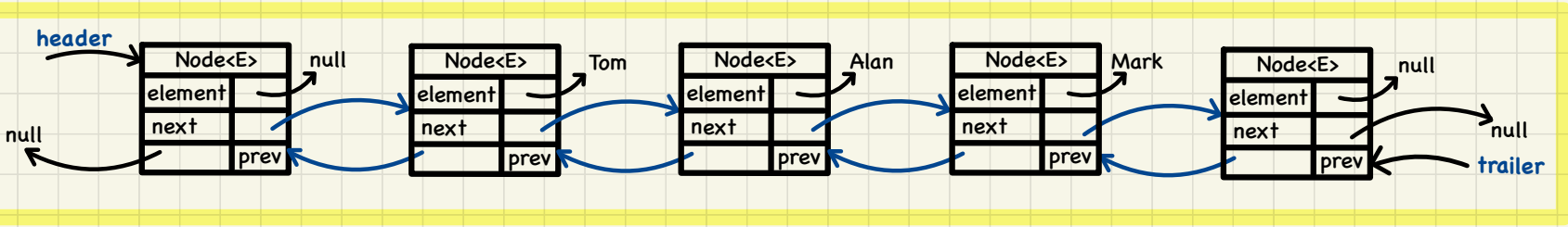


Generic DLL in Java: Removing from the Middle

```java
@Test
public void test_String_DLL_removeAt() {
    DoublyLinkedList<String> list = new DoublyLinkedList<>();
    list.addFirst("Mark");
    list.addFirst("Alan");
    list.addFirst("Tom");
    assertTrue(list.getSize() == 3);
    list.removeAt(1);
    assertTrue(list.getSize() == 2);
    list.removeAt(0);
    assertTrue(list.getSize() == 1);
    list.removeAt(0);
    assertTrue(list.getSize() == 0);
}
```

Execute

```java
removeAt (int i) {
  if (i < 0 || i >= size) {
    throw new IllegalArgumentExce
  else {
    Node<E> node = getNodeAt(i);
    remove (node);
  }
}
```



Lecture 13 - February 25

General Trees

Linear vs. Non-Linear Structures
General Trees: Terminology
Generic TreeNode in Java

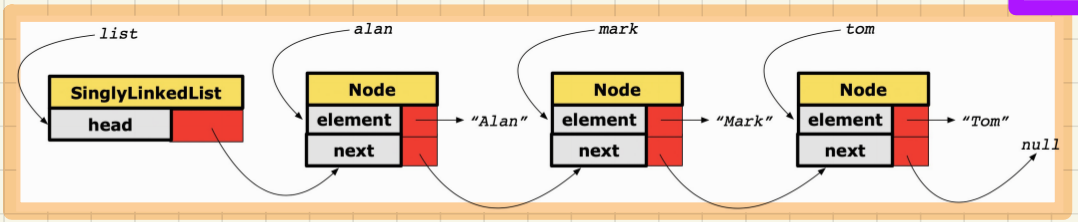
Announcements/Reminders

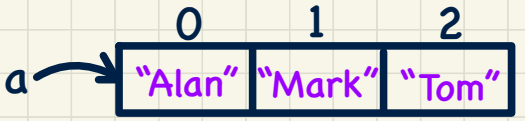
- Survey on **Makeup Lecture** for ProgTest1
- **Assignment 3** (on **linked** **Trees**) to be released
- ~~**ProgTest2** (on **linked** **Trees**) to be released~~
- This week's office hour: 3pm, Wed

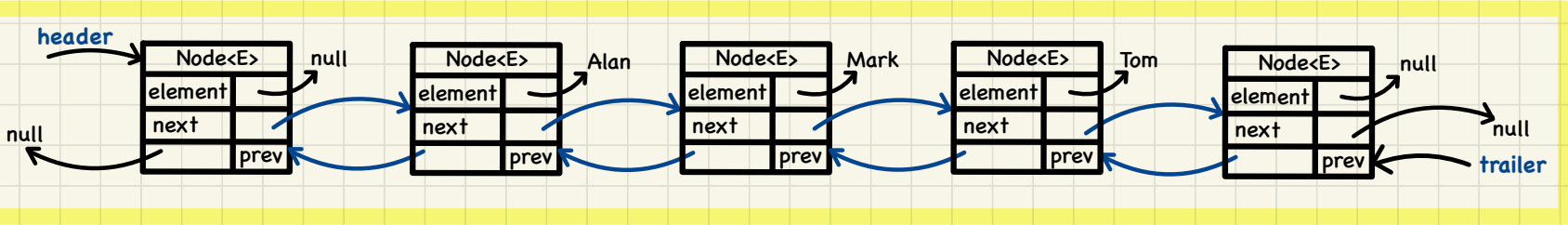
Running Time: Arrays vs. SLL vs. DLL

see earlier discussion

Operation \ Data Structure		Array	Singly-Linked List	Doubly-Linked List
size				O(1)
first/last element				O(1)
element at index i		O(1)	O(n)	O(n)
remove last element				O(1)
add/remove first element, add last element		O(n)	O(1)	O(1)
add/remove i^{th} element	given reference to $(i-1)^{th}$ element			O(1)
	not given			O(n)







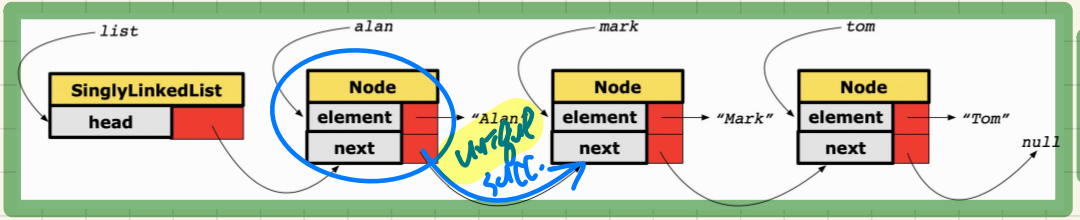
Trees

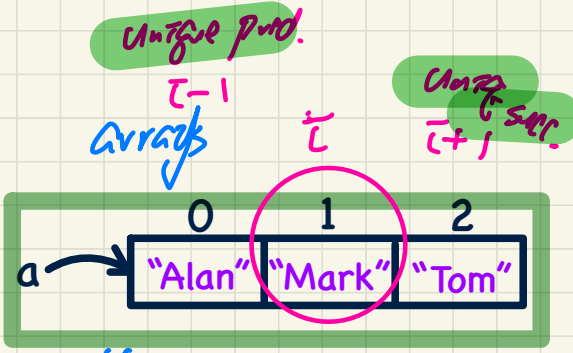
a.
1. General Trees
2. Binary Trees (BTs)

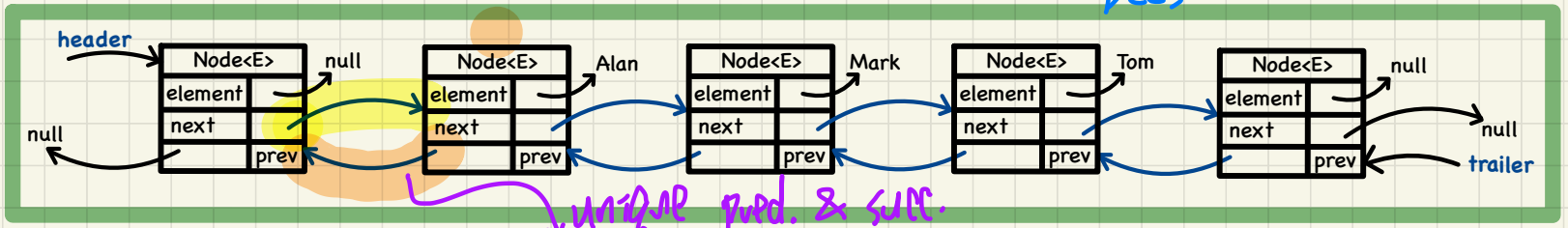
b.
3. Binary Search Trees (BSTs)

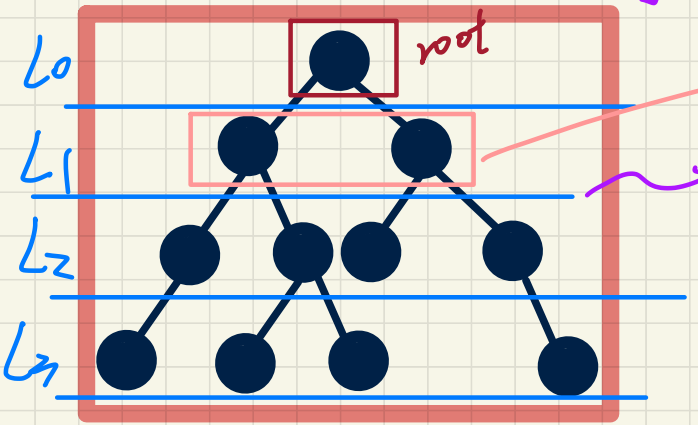
c.
4. Balanced BSTs
5. Priority Queues
6. Heap
7. Heap Sort

Linear vs. Non-Linear Structures





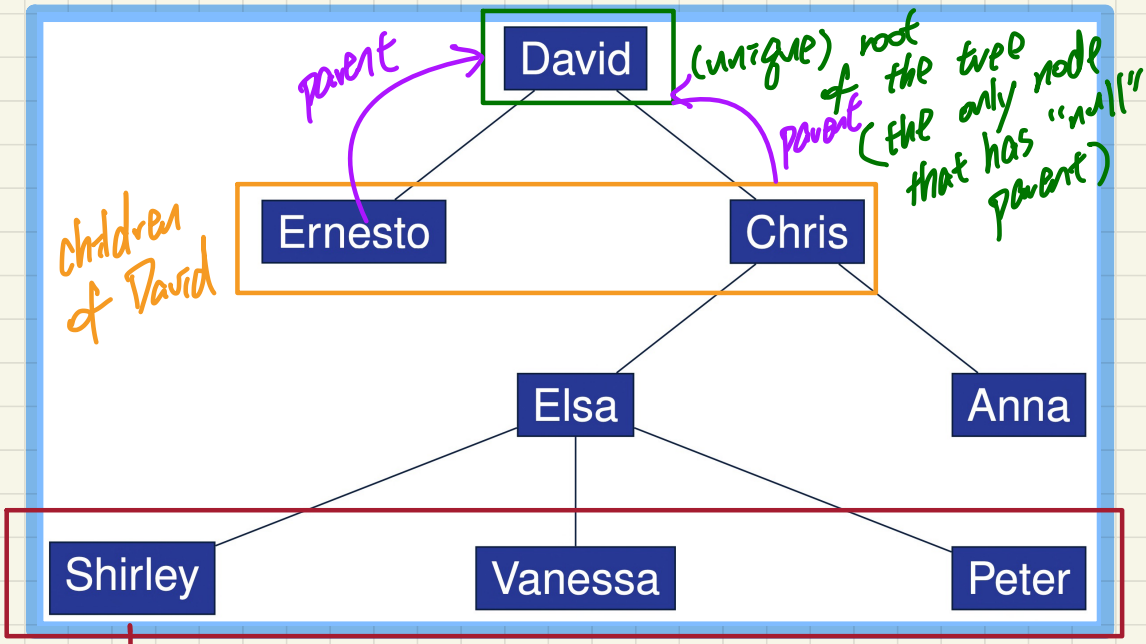




multiple "successors" of the root (which can be stored in a linear DS)

hierarchical structure (levels)

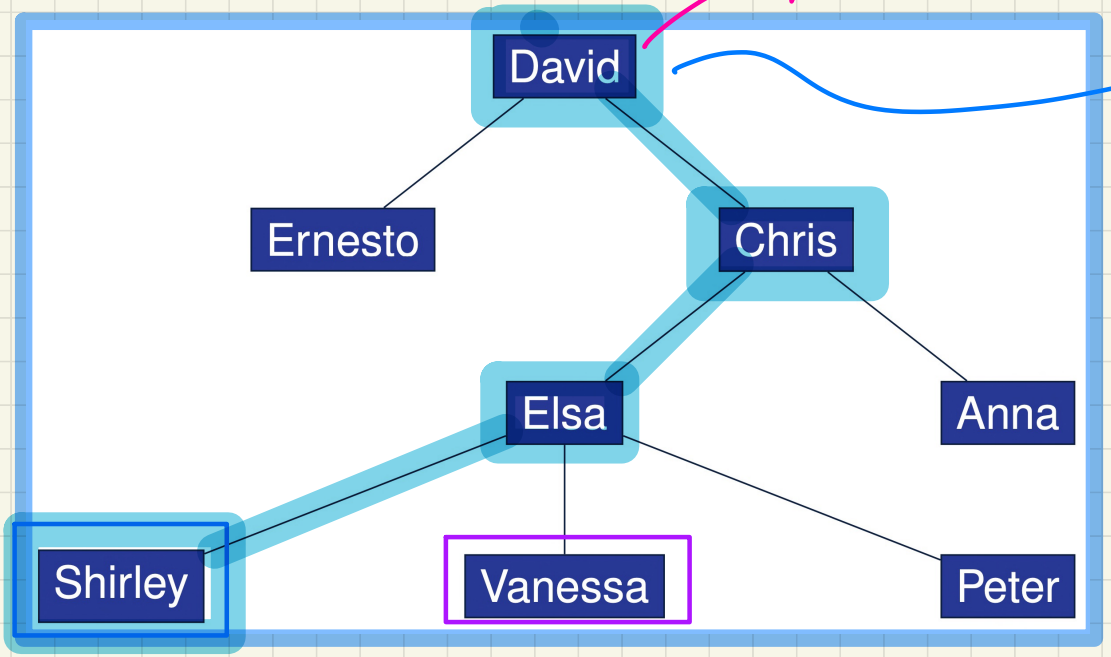
General Trees: Terminology (1)



- root
- parent
- children
- ancestors
- descendants
- siblings

↓

nodes sharing the same parent



root has the most # of descendants (all nodes).

(Shirley, Elsa, Chris, David) ancestor path of Shirley

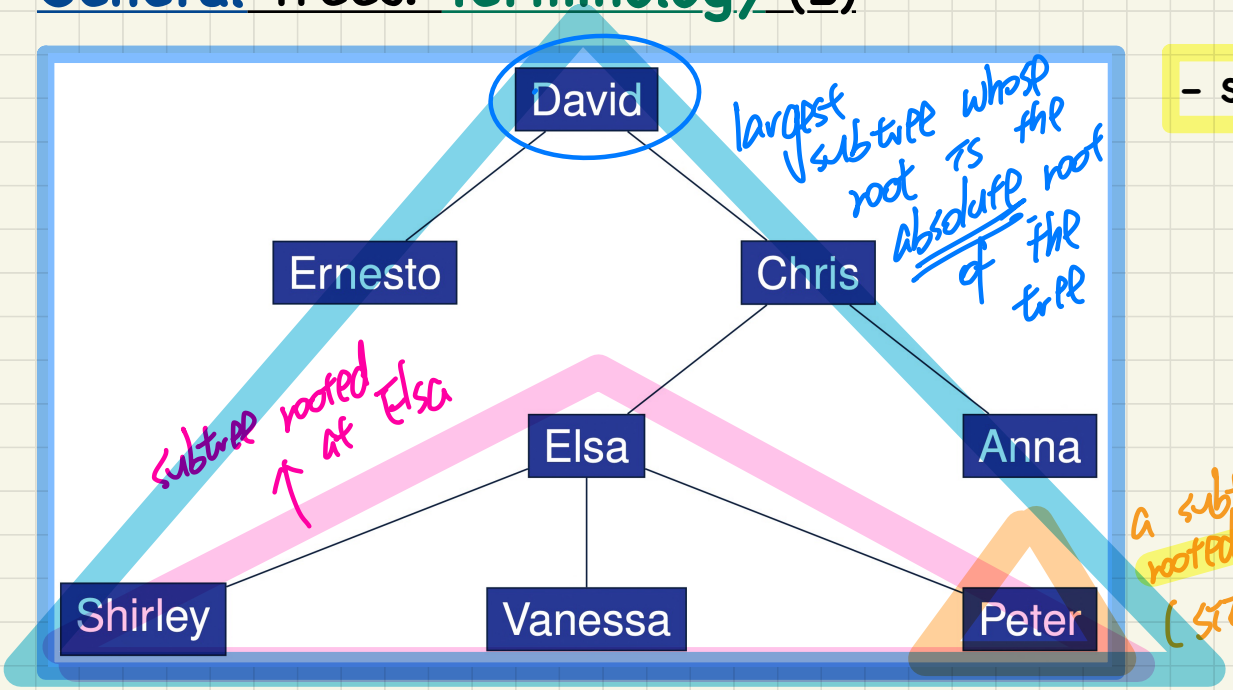
unique ∴ each node has unique parent.

Ancestors of Shirley: S, Elsa, C, D

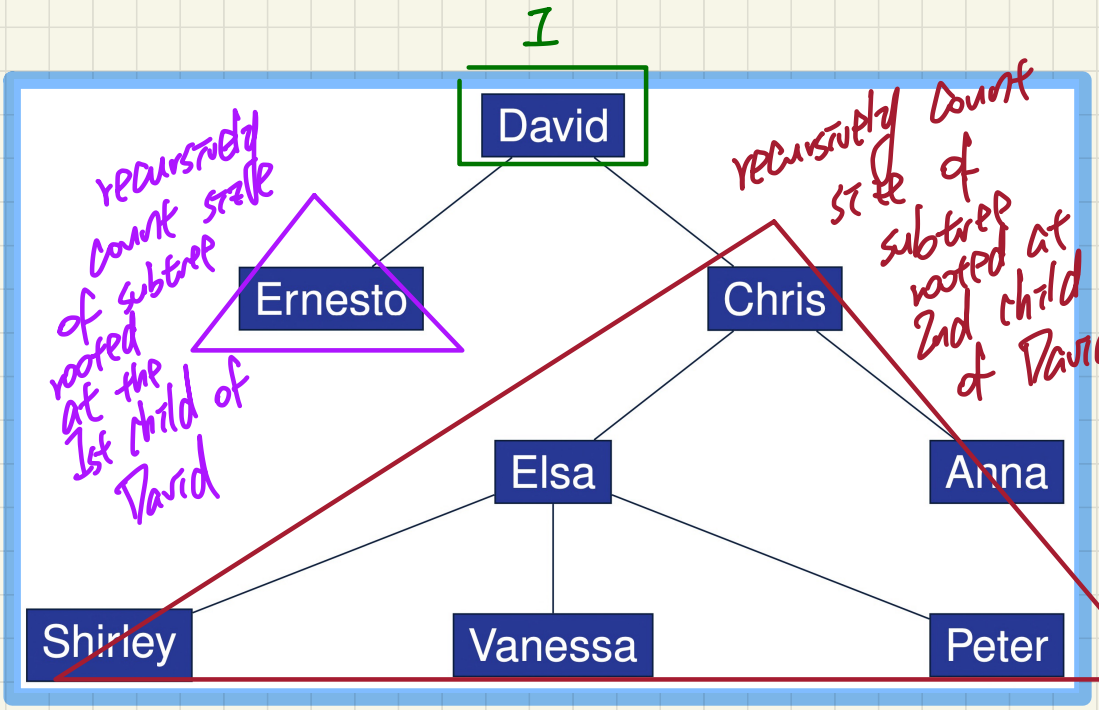
Descendants of Vanessa: Vanessa

A node is both its ancestor and descendant.

General Trees: Terminology (2)



- subtree



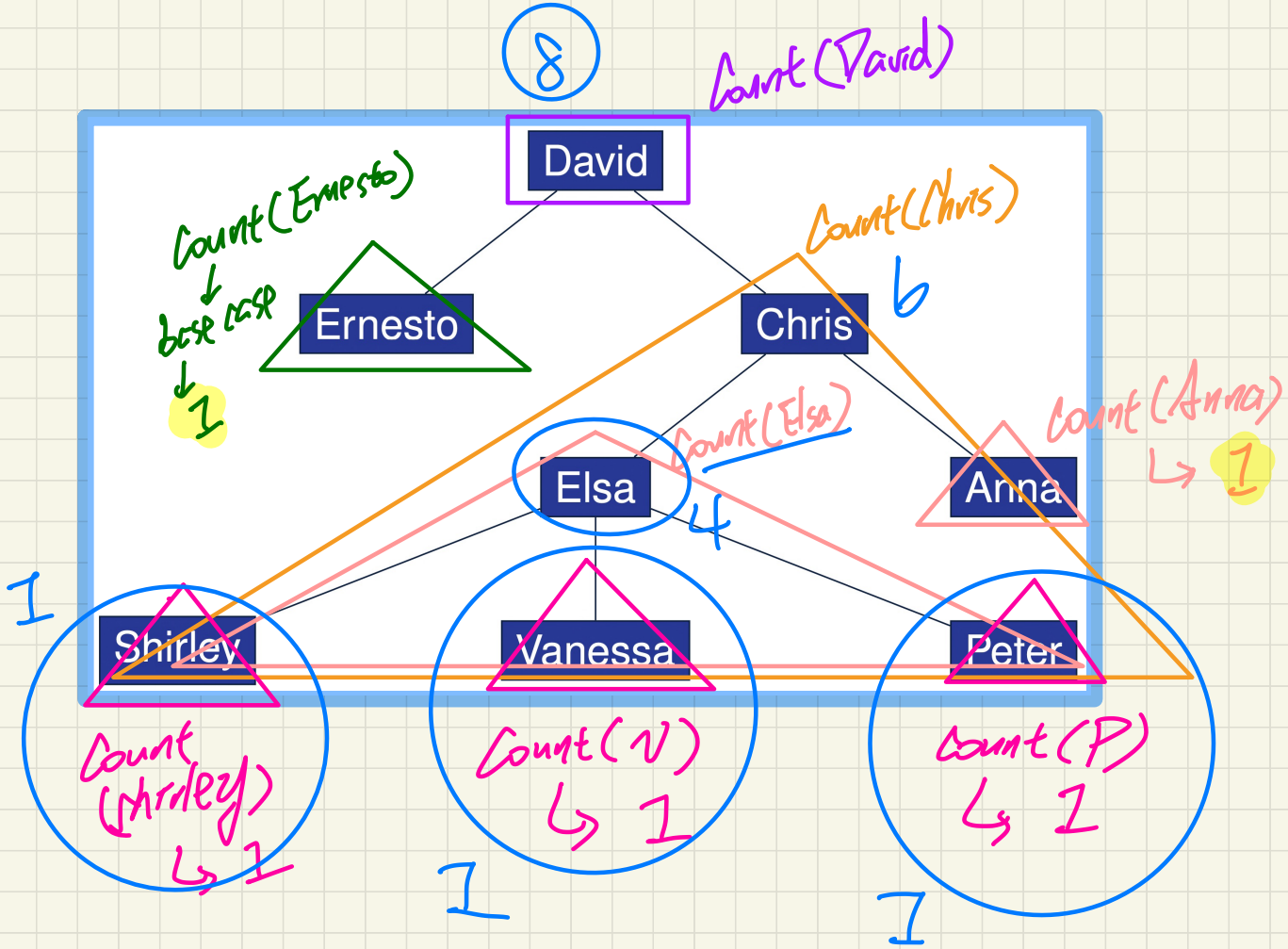
Problem

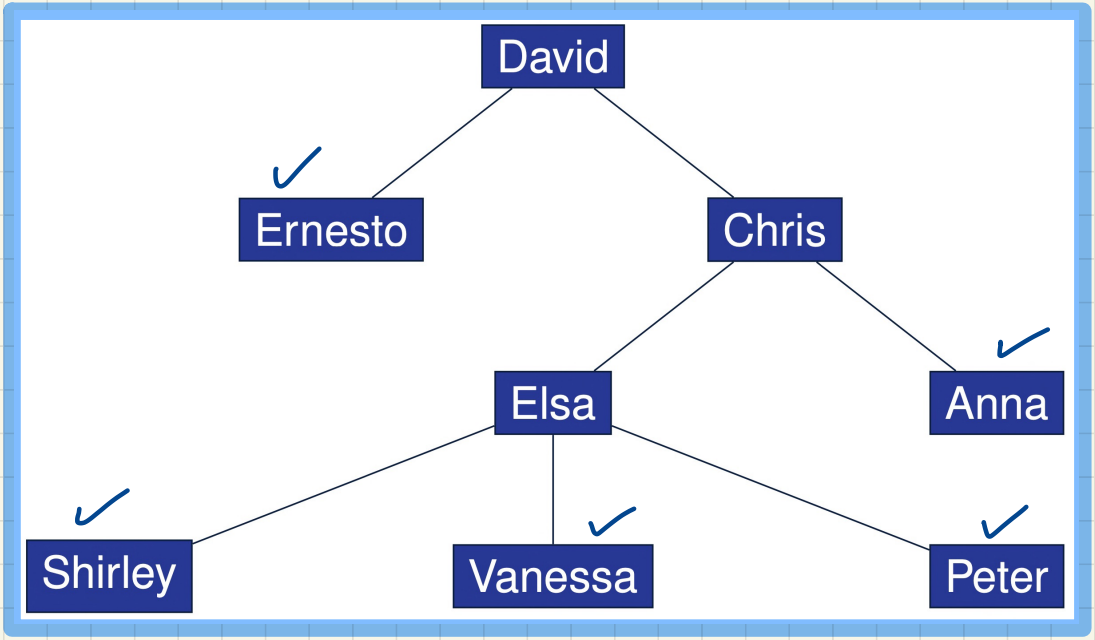
Count the # of nodes in a tree rooted at n.

Count(David)

↓ root of some subtree.

- subtrees are the subproblems for you to make recursive calls on.


8
Count(David)
David
Count(Ernesto)
base case
1
Ernesto
Count(Chris)
6
Chris
Count(Elsa)
4
Elsa
Count(Anna)
1
Anna
1
Shirley
Vanessa
Peter
Count(Shirley)
1
Count(V)
1
Count(P)
1
1
1

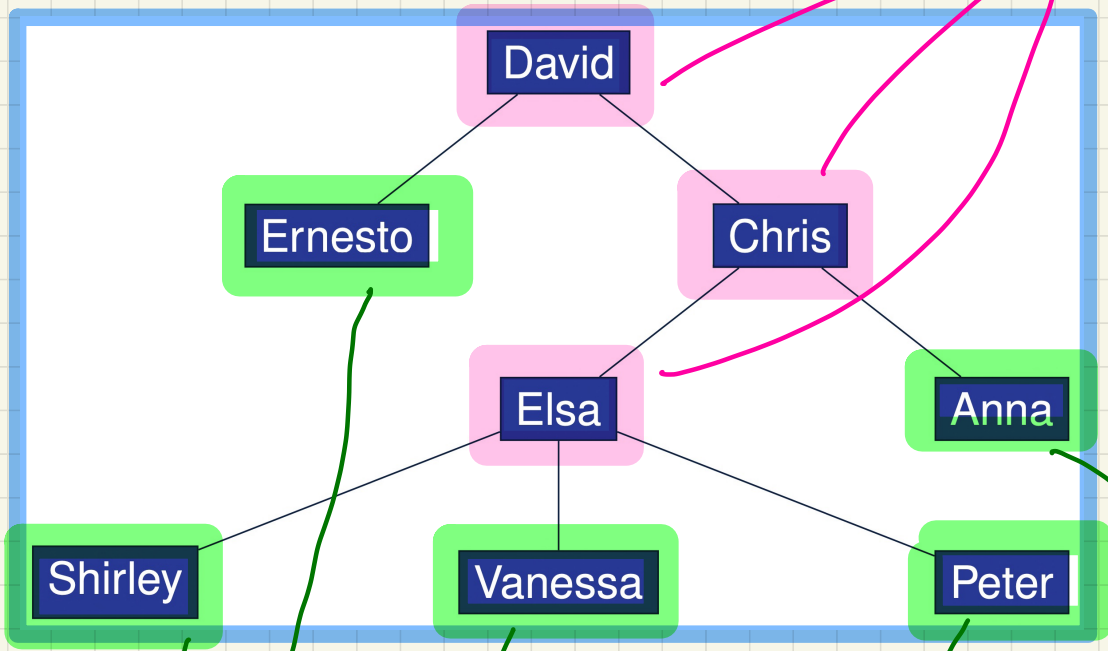


How many subtrees in total?

f = # nodes in the tree.

size	# subtrees
1	5
⋮	
f	1 rooted at David

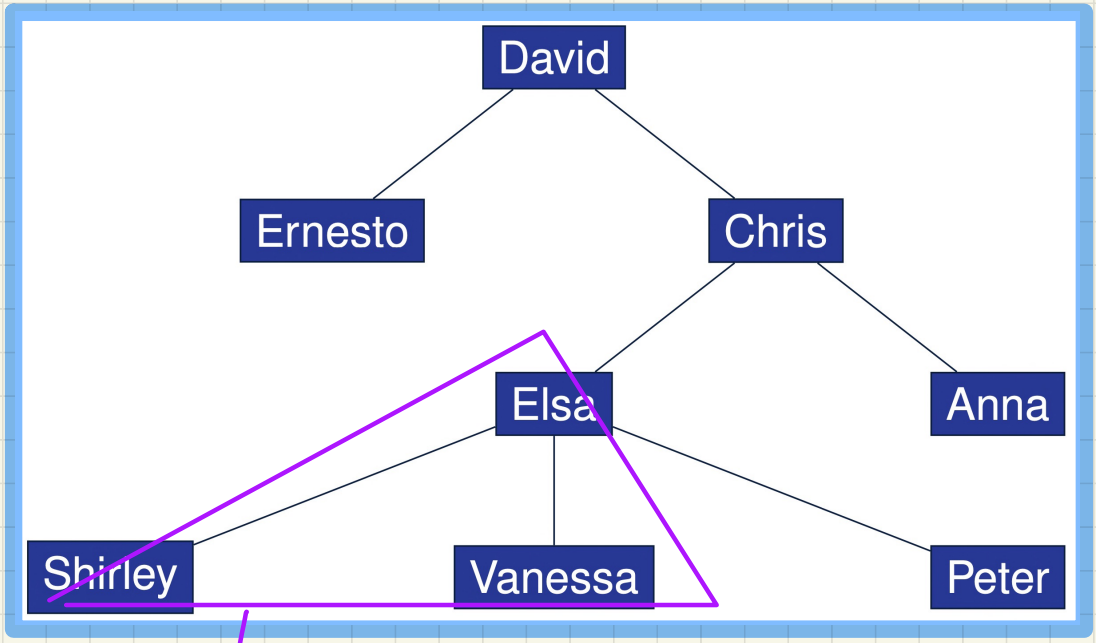
General Trees: Terminology (3)



internal nodes → recursive cases

- external nodes
- internal nodes

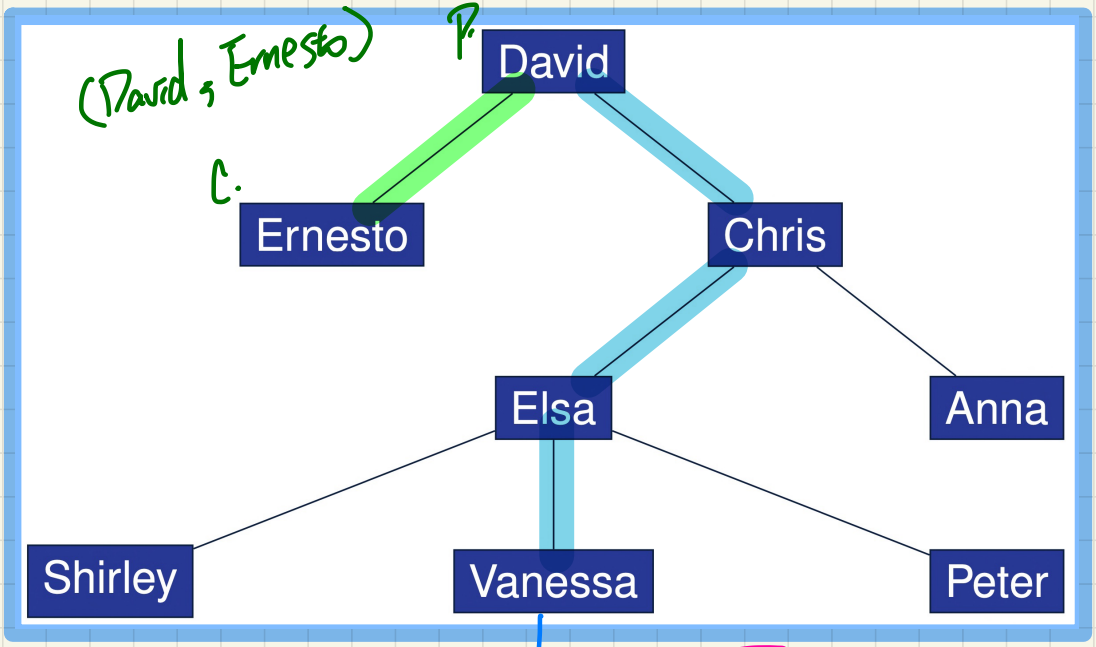
external nodes (leaves) → base cases.



a subtree rooted at Elsa?

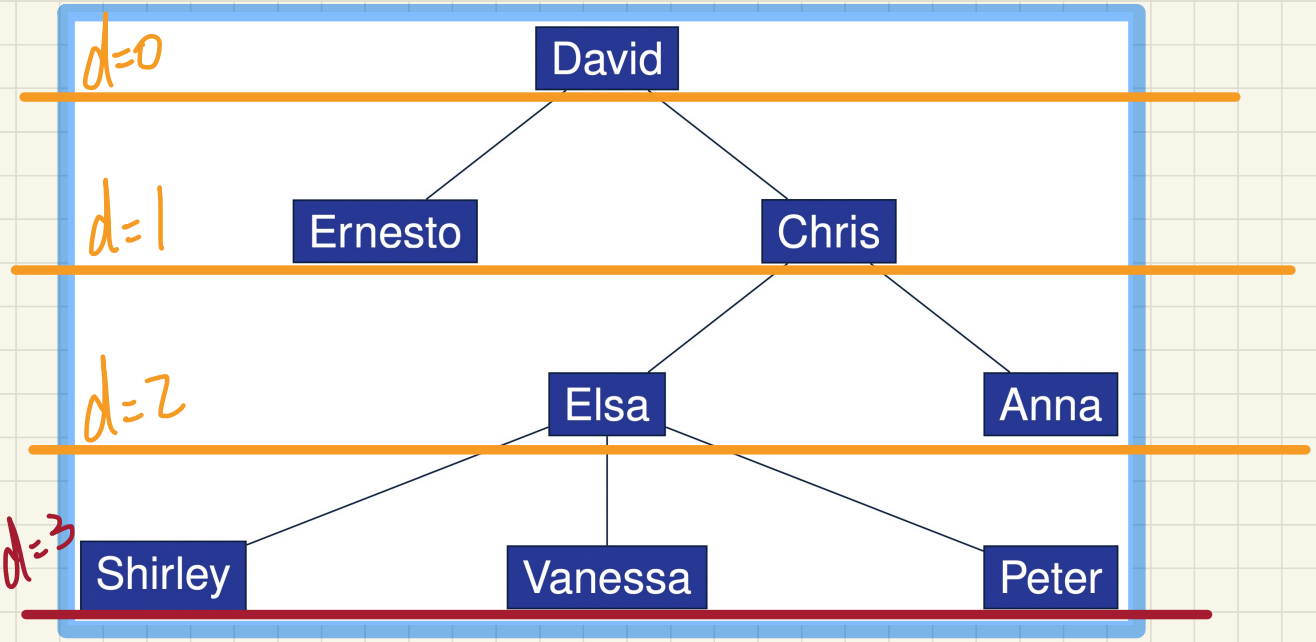
No :(Peter not included

General Trees: Terminology (4)



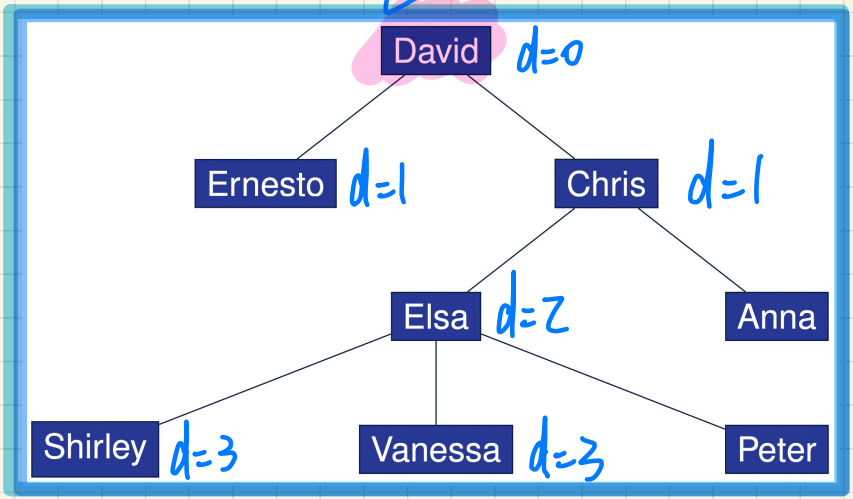
- edge
- path
- depth
- height

path (V, E, C, D) → 3 edges in this path



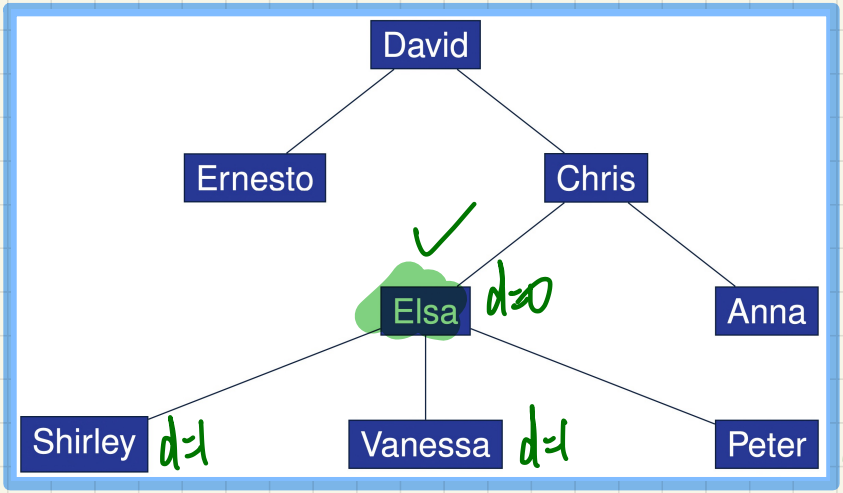
depth of a node : # of edges (parent links) from node to root.

height of a (sub-)tree : max depth of descendants



Height of tree rooted at David:

max d value: 3

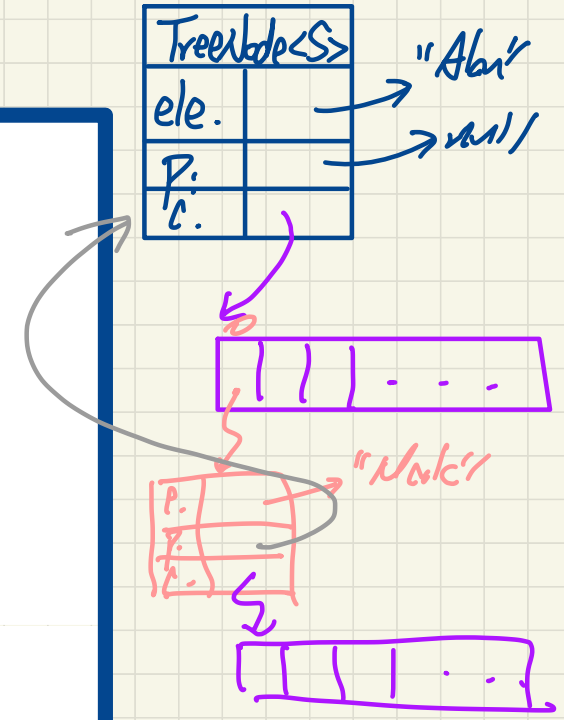


Height of tree rooted at Elsa:

max d value: 1

Generic, General Tree Nodes

```
public class TreeNode<E> {
  private E element; /* data object */
  private TreeNode<E> parent; /* unique parent node */
  private TreeNode<E>[] children; /* list of child nodes */

  private final int MAX_NUM_CHILDREN = 10; /* fixed max */
  private int noc; /* number of child nodes */

  public TreeNode(E element) {
    this.element = element;
    this.parent = null;
    this.children = (TreeNode<E>[])
      Array.newInstance(this.getClass(), MAX_NUM_CHILDREN);
    this.noc = 0;
  }
  public E getElement() { ... }
  public TreeNode<E> getParent() { ... }
  public TreeNode<E>[] getChildren() { ... }

  public void setElement(E element) { ... }
  public void setParent(TreeNode<E> parent) { ... }
  public void addChild(TreeNode<E> child) { ... }
  public void removeChildAt(int i) { ... }
}
```



Compare:

+ prev ref.
+ next ref.

in a DLN.

Lecture 14 - March 4

General Trees, Binary Trees

Initializing a Generic Array
Recursive Definitions of (Binary) Trees
Trees in Java: Construction, Depth

Announcements/Reminders

- ProgTest1 results to be released by Friday, Mar 14
- Makeup Lecture (on ADTs, Stacks) posted
- WrittenTest guide and example questions to be release
- Assignment 3 (on linked Trees) to be released
- Lecture notes template, Office Hours, TA Contact

General Trees: Recursive Definition

- root
- size

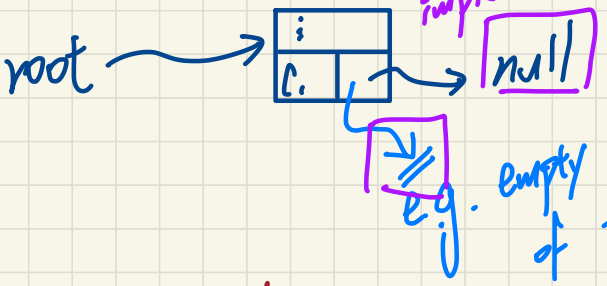
Case 0: Empty Tree

ϕ

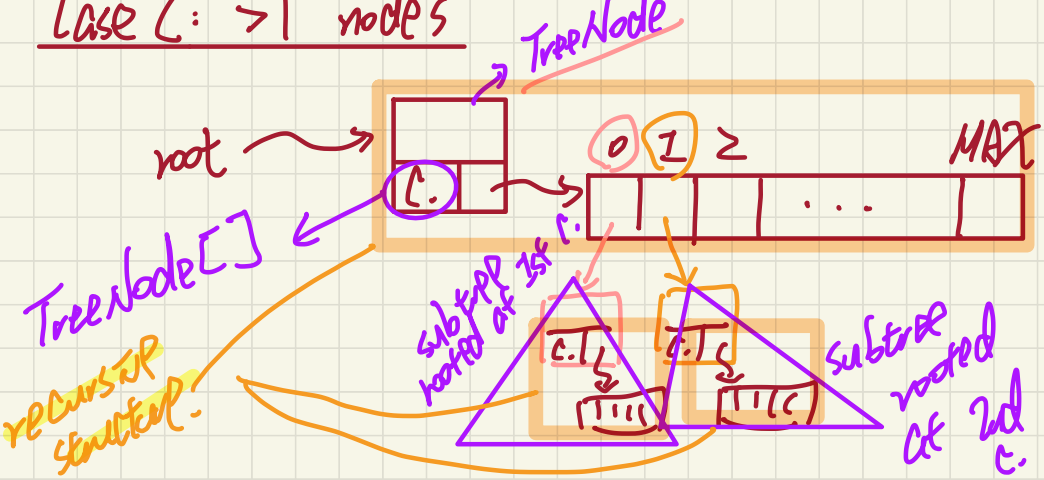
root → null

size == 0

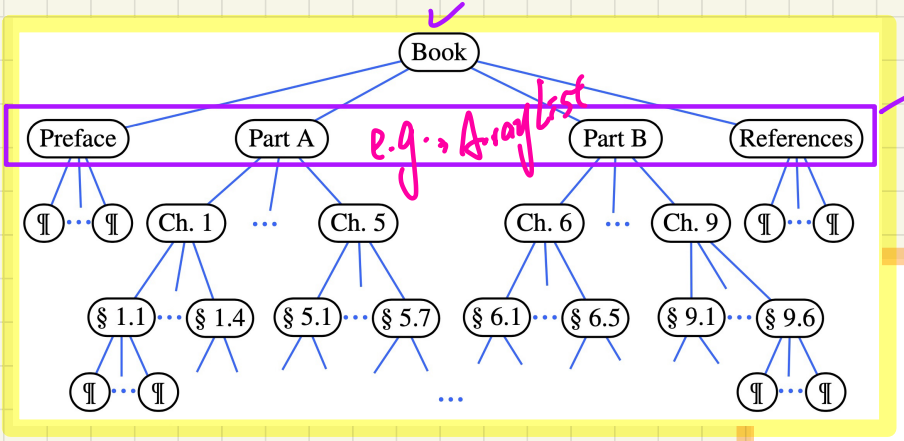
Case 1: A singleton tree

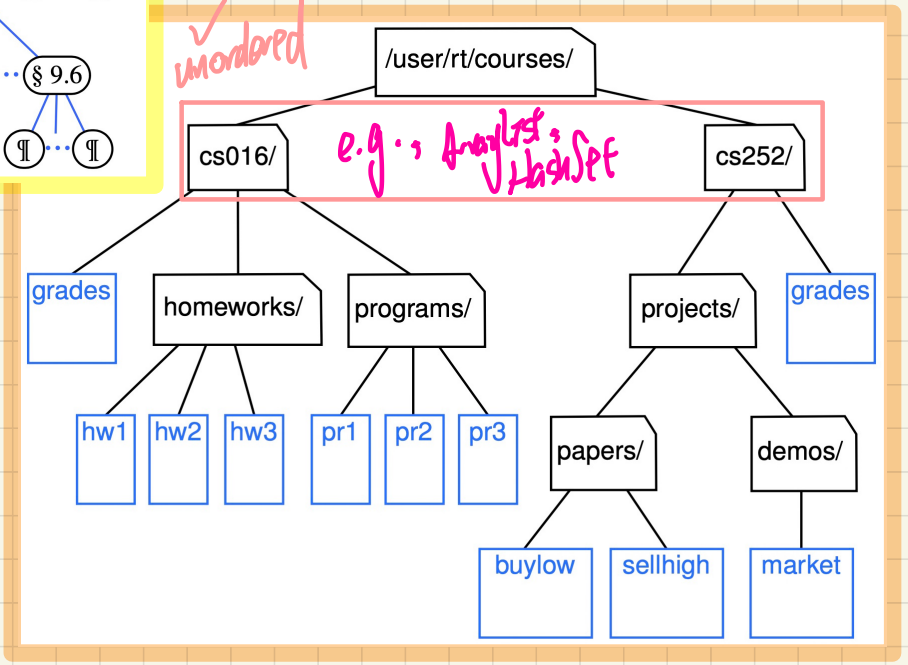


Case 2: >1 nodes



General Trees: Ordered vs. Unordered Trees





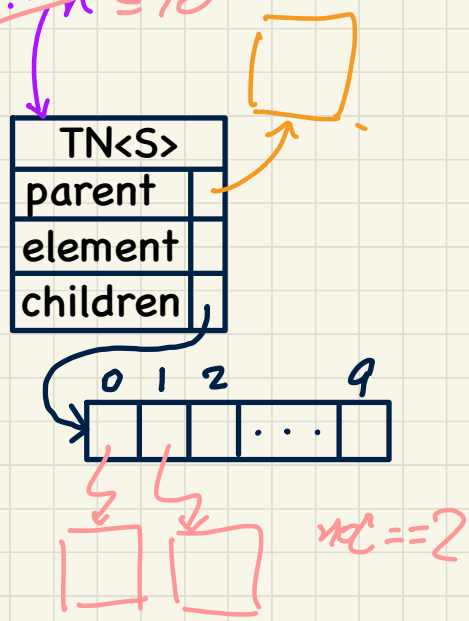
red
↳ tree

Generic, General Tree Nodes

```
public class TreeNode<E> {
  private E element; /* data object */
  private TreeNode<E> parent; /* unique parent node */
  private TreeNode<E>[] children; /* list of child nodes */

  private final int MAX_NUM_CHILDREN = 10; /* fixed max */
  private int noc; /* number of child nodes */

  public TreeNode(E element) {
    this.element = element;
    this.parent = null;
    this.children = (TreeNode<E>[])
      Array.newInstance(this.getClass(), MAX_NUM_CHILDREN);
    this.noc = 0;
  }
  public E getElement() { ... }
  public TreeNode<E> getParent() { ... }
  public TreeNode<E>[] getChildren() { ... }

  public void setElement(E element) { ... }
  public void setParent(TreeNode<E> parent) { ... }
  public void addChild(TreeNode<E> child) { ... }
  public void removeChildAt(int i) { ... }
}
```

n.children.length is 10

vs. n.noc, $n.noc \le 10$

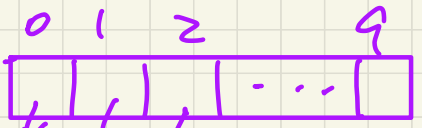


Compare:

+ prev ref.
+ next ref.

in a DLN.

Instantiating Generic Structures in Java




```
class ArrayStack<E> {
  private E[] data;
  ...
  public ArrayStack<E>() {

  }
}
```

data[i] = 23;
data[i] = "alan"

data = new E[10];

data = (E[]) new Object[10];

```
class TreeNode<E> {
  private TreeNode<E>[] children;
  ...
  public TreeNode<E>() {

  }
}
```

E is wrapped within the TreeNode class

workaround just for the runtime. static type of data[i] remains E

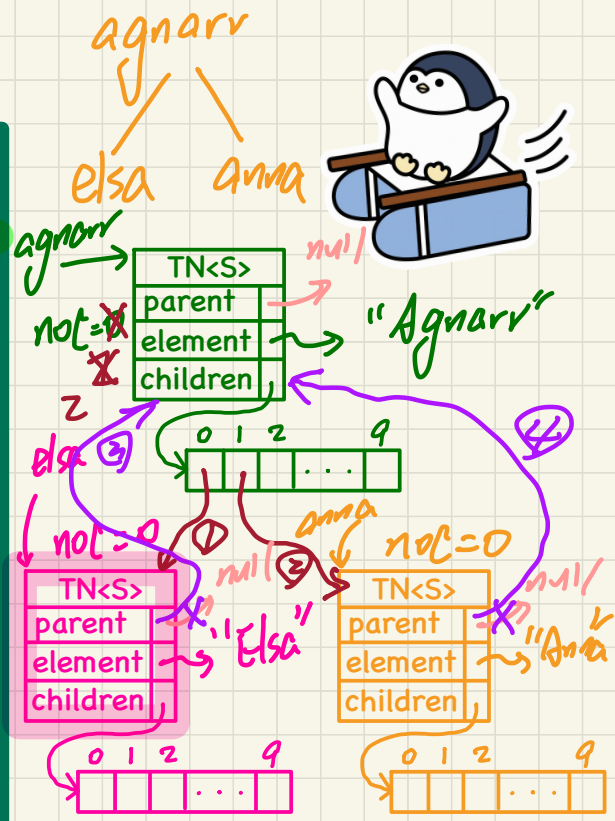
children = (TN<E>) new Object[10];

children = (TN<E>) Array.newInstance(this.getClass(), 10);

① this.getClass() → TreeNode<E>.getClass()

Tracing: Constructing a Tree

```java
@Test
public void test_general_trees_construction() {
  TreeNode<String> agnarr = new TreeNode<>("Agnarr");
  TreeNode<String> elsa = new TreeNode<>("Elsa");
  TreeNode<String> anna = new TreeNode<>("Anna");

  agnarr.addChild(elsa);
  agnarr.addChild(anna);
  elsa.setParent(agnarr);
  anna.setParent(agnarr);

  assertNull(agnarr.getParent());
  assertTrue(agnarr == elsa.getParent());
  assertTrue(agnarr == anna.getParent());
  assertTrue(agnarr.getChildren().length == 2);
  assertTrue(agnarr.getChildren()[0] == elsa);
  assertTrue(agnarr.getChildren()[1] == anna);
}
```



Aliasing ① elsa

② agnarr.getChildren()[0]

③ elsa.getParent().getChildren[0]

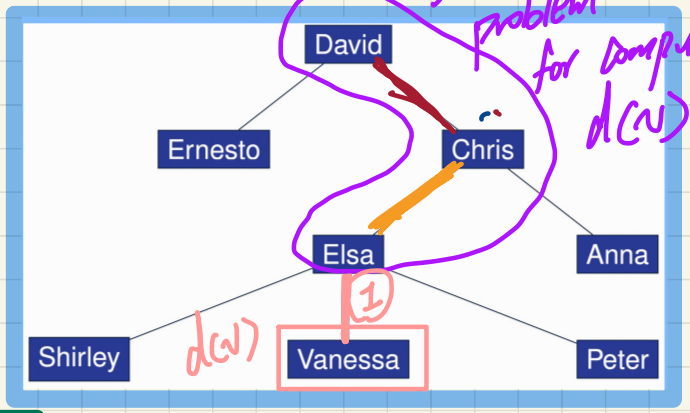
Tracing: Computing a Node's Depth

root of some subtree

```
public int depth(TreeNode<E> n) {
  if(n.getParent() == null) {
    return 0;
  }
  else {
    return 1 + depth(n.getParent());
  }
}
```

depth of root is 0

the depth of v's parent



strictly smaller problem for computing d(v)

```
@Test
public void test_general_trees_depths() {
  ... /* constructing a tree as shown above */
  TreeUtilities<String> u = new TreeUtilities<>();
  assertEquals(0, u.depth(david));
  assertEquals(1, u.depth(ernesto));
  assertEquals(1, u.depth(chris));
  assertEquals(2, u.depth(elsa));
  assertEquals(2, u.depth(anna));
  assertEquals(3, u.depth(shirley));
  assertEquals(3, u.depth(vanessa));
  assertEquals(3, u.depth(peter));
}
```

depth(vanessa)

.getP != null

= 1 + depth(Elsa)

.getP != null

= 1 + 1 + depth(Chris)

.getP != null

= 1 + 1 + 1 + depth(David)

0

= 3

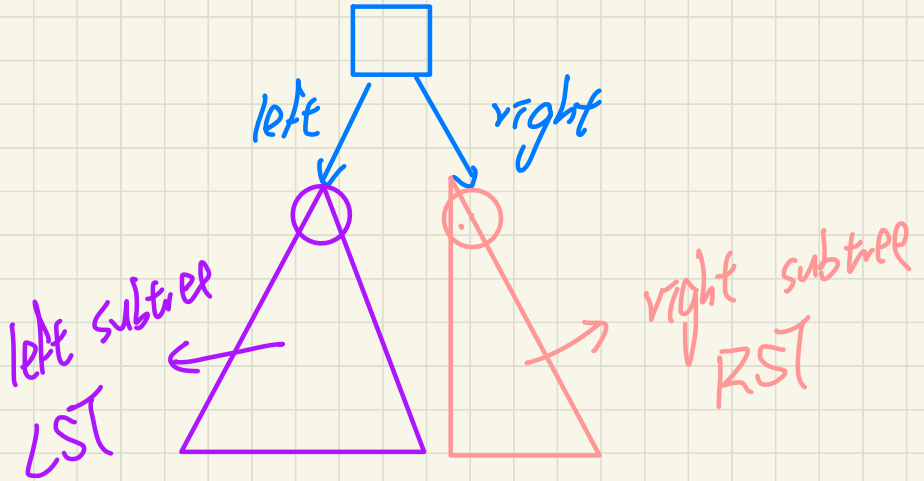
Binary Trees: Recursive Definition

- root
- size

Case 0: Empty Tree

Case 1: A singleton Tree

Case 2: >= 2 nodes



Deriving the Sum of a Geometric Sequence

Initial Term: I

Common Factor: r

Number of Terms: k

$1 + 2 + 4 + 8 + 16 + \cdots + 1024$ ($1 = 2^0$, $2 = 2^1$, $4 = 2^2$, $1024 = 2^{10}$)

$*2$ $*2$

$k = 11$.

$$S_k = \underbrace{I}_{I \cdot r^0} + I \cdot r + I \cdot r^2 + I \cdot r^3 + \cdots + I \cdot r^{k-1}$$

$$r \cdot S_k = I \cdot r + I \cdot r^2 + I \cdot r^3 + \cdots + I \cdot r^{k-1} + I \cdot r^k$$

$$r \cdot S_k - S_k = (r-1) \cdot S_k = I \cdot r^k - I = I \cdot (r^k - 1)$$

$2^k - 1$?!?!

$$S_k = \frac{I \cdot (r^k - 1)}{r - 1}$$

For BT: $r = 2$, $I = 1$

$S_k = r^k - 1$ → depth of "bottom" node

root

Lecture 15 - March 6

Binary Trees

Binary Trees: Math Properties
Tree Traversals

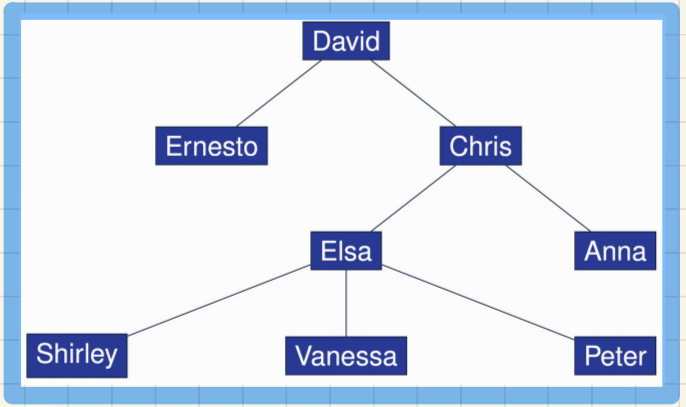
Announcements/Reminders

- Assignment 3 (on linked Trees) released
- WrittenTest
  + guide released
  + example questions to be release
- Makeup Lecture (on ADTs, Stacks) posted
- Lecture notes template, Office Hours, TA Contact

Tracing: Computing a Tree's Height

```
public int height(TreeNode<E> n) {
  TreeNode<E>[] children = n.getChildren();
  if(children.length == 0) { return 0; }
  else {
    int max = 0;
    for(int i = 0; i < children.length; i ++) {
      int h = 1 + height(children[i]);
      max = h > max ? h : max;
    }
    return max;
  }
}
```

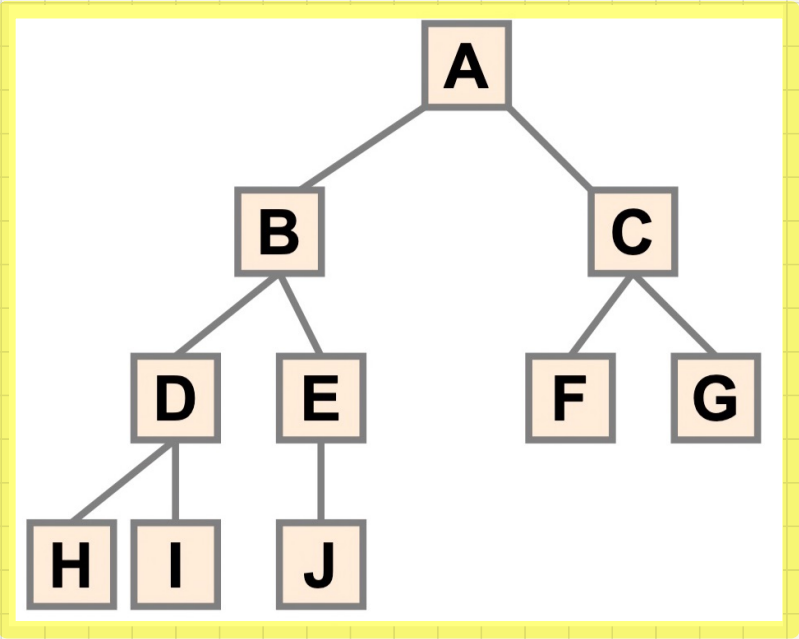
not



height(chris)

```
@Test
public void test_general_trees_heights() {
  ... /* constructing a tree as shown above */
  TreeUtilities<String> u = new TreeUtilities<>();
  /* internal nodes */
  assertEquals(3, u.height(david));
  assertEquals(2, u.height(chris));
  assertEquals(1, u.height(elsa));
  /* external nodes */
  assertEquals(0, u.height(ernesto));
  assertEquals(0, u.height(anna));
  assertEquals(0, u.height(shirley));
  assertEquals(0, u.height(vanessa));
  assertEquals(0, u.height(peter));
}
```

BT Terminology: LST vs. RST



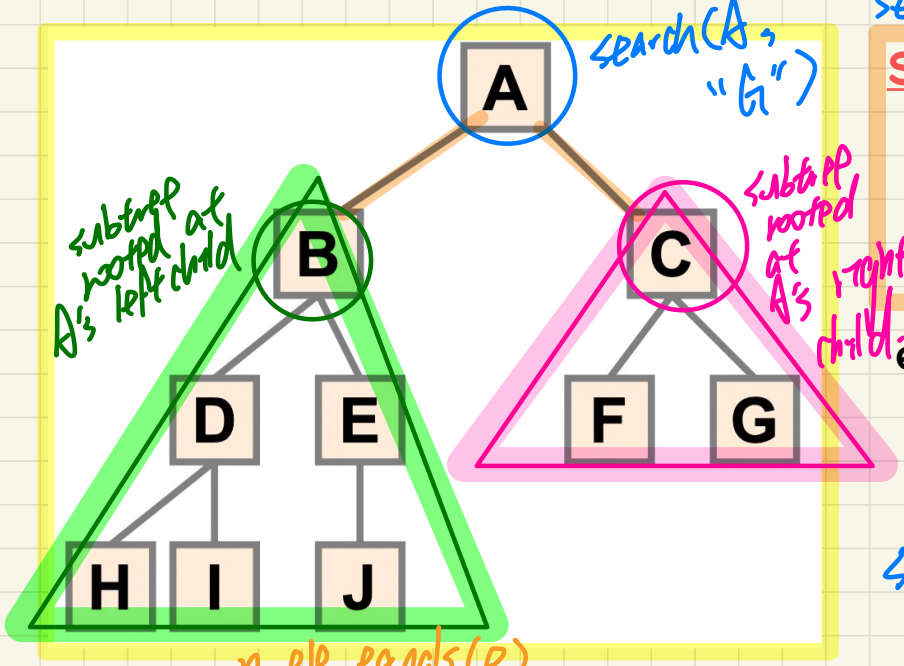
Strategy of Recursion on BT:

+ Do something on root
+ Recur on LST
+ Recur on RST

e.g.,

+ counting size

BT Terminology: LST vs. RST



search(A, "G") → T
search(A, "Z") → F

Strategy of Recursion on BT:

+ Do something on root
+ Recur on LST
+ Recur on RST

e.g.,

+ searching item

search(n, e) n: tree node, e: data item

↳ if n is external → n.data.equals(e)

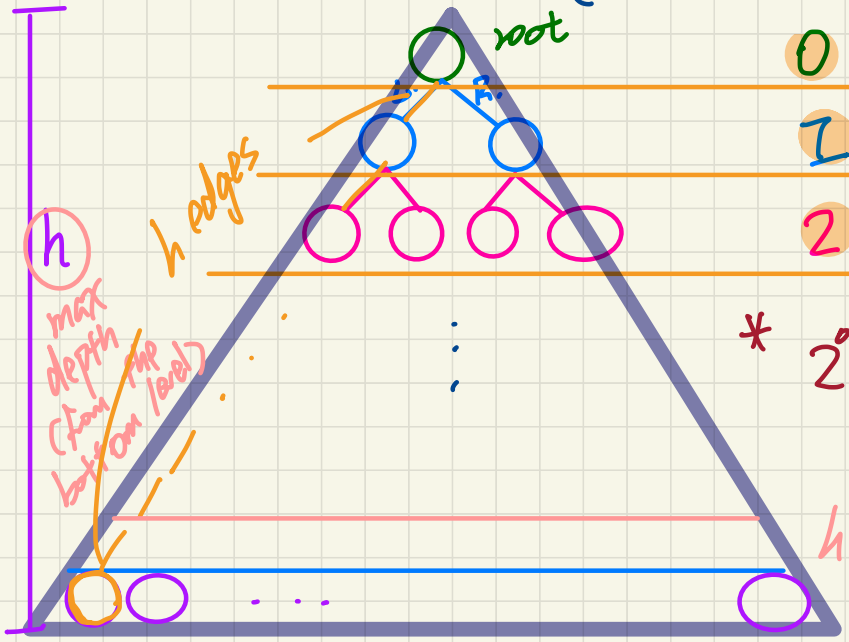
else /* n is internal */

n.ele.equals(e)
||
return search(A.left, e)
||
search(A.right, e)

BT Terminology: Depths, Levels, Max # of Nodes

$S_k = \frac{I \cdot (r^k - 1)}{r - 1}$ BT: $\frac{1 \cdot (2^k - 1)}{2 - 1} = 2^k - 1$ (k = height)

* Max # of nodes in BT with height h



BT structure

Level ? (d)

Max # Nodes at Level ?

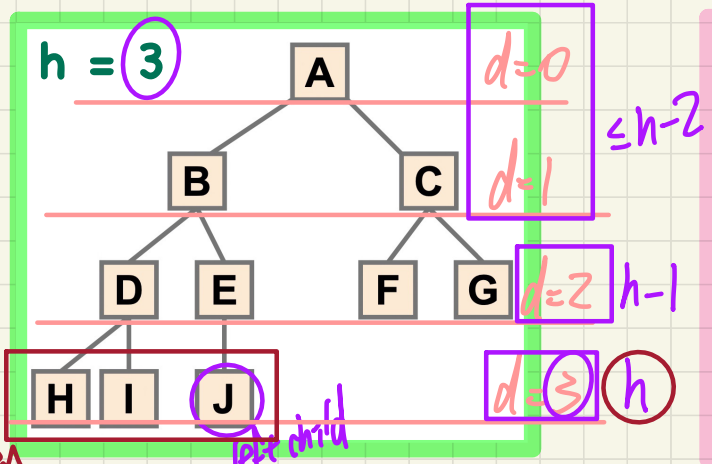
Level	Max # Nodes at Level
0	$1 = 2^0$
1	$2 = 2^1$
2	$4 = 2^2$
h-1	2^{h-1}
h	2^h

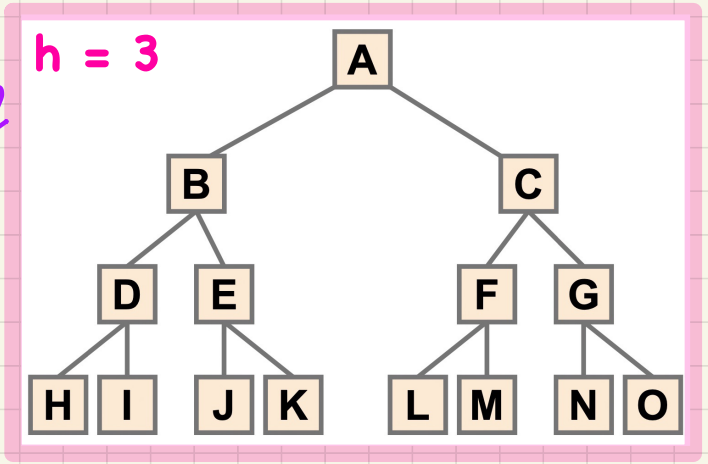
* $2^0 + 2^1 + 2^2 + \dots + 2^{h-1} + 2^h = 2^{h+1} - 1$

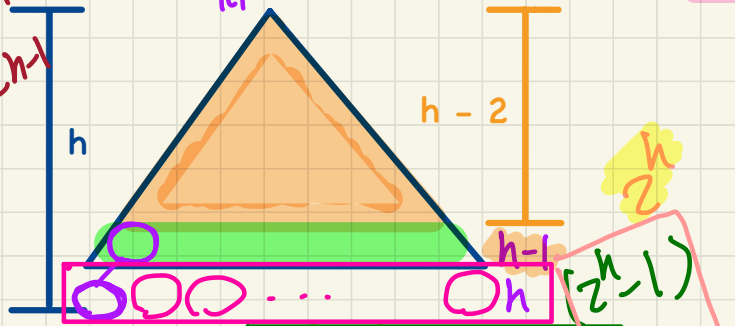
Exercise: Max # of nodes from level 0 to level h-1?

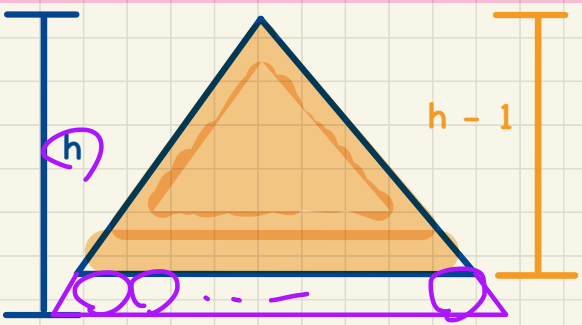
BT Terminology: Complete vs. Full BTs

$\sum_k = 2^k - 1$







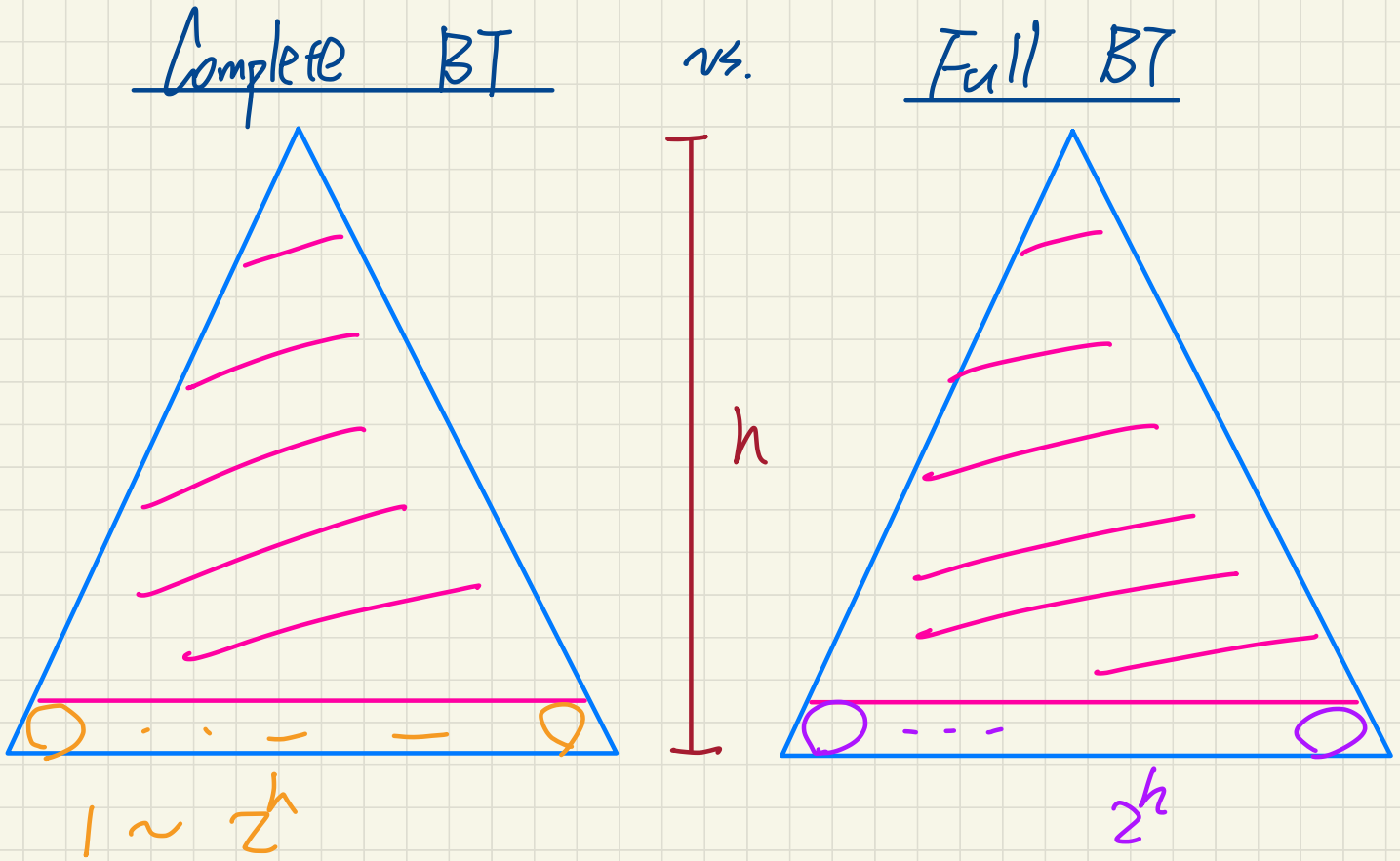


Min # nodes? $(2^0+2^1+\cdots+2^{h-1})+1$

Max # nodes? $(2^0+2^1+\cdots+2^{h-1})+2^h$ $\quad 2^{h+1}-1$

Min # nodes?

Max # nodes? $2^0+2^1+\cdots+2^h = 2^{h+1}-1$


Complete BT
vs.
Full BT
h
1 ~ 2^h
2^h

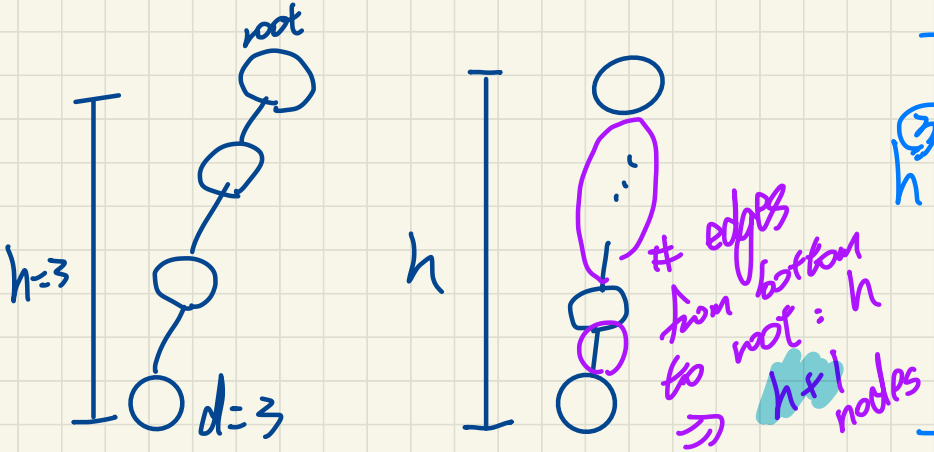
BT Properties: Bounding # of Nodes

Given a *binary tree* with ***height*** h, the ***number of nodes*** n is bounded as:

$$h + 1 \leq n \leq 2^{h+1} - 1$$

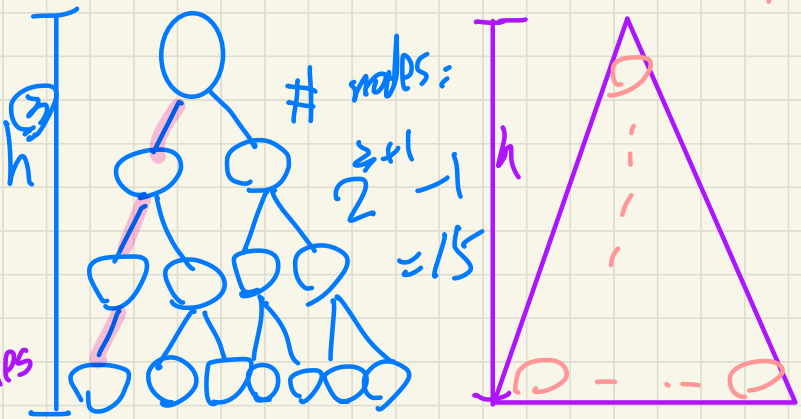
For example, say h = 3

Minimum # of nodes



Maximum # of nodes

$$2^0 + 2^1 + \dots + 2^h = 2^{h+1} - 1$$



BT Properties: Bounding Height of Tree

$\log 2^{h+1} = h+1$

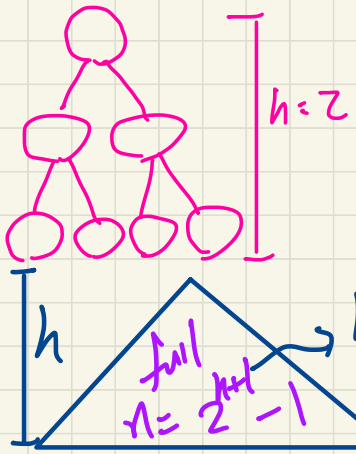
$\log 2^3 = 3$

Given a *binary tree* with *n* **nodes**, the ***height*** *h* is bounded as:

$$\log(n+1) - 1 \le h \le n - 1$$

For example, say n = 7

Minimum height



h is min if the BT is full.

min height $= \log(7+1) - 1$

$= 2$

$n = 2^{h+1} - 1$

$n + 1 = 2^{h+1}$

$\log(n+1) = h+1$

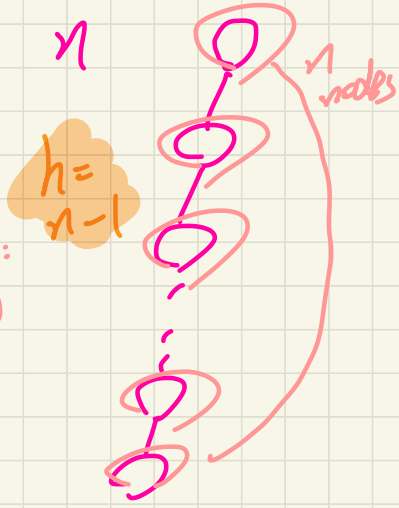
Maximum height

$n = 7$

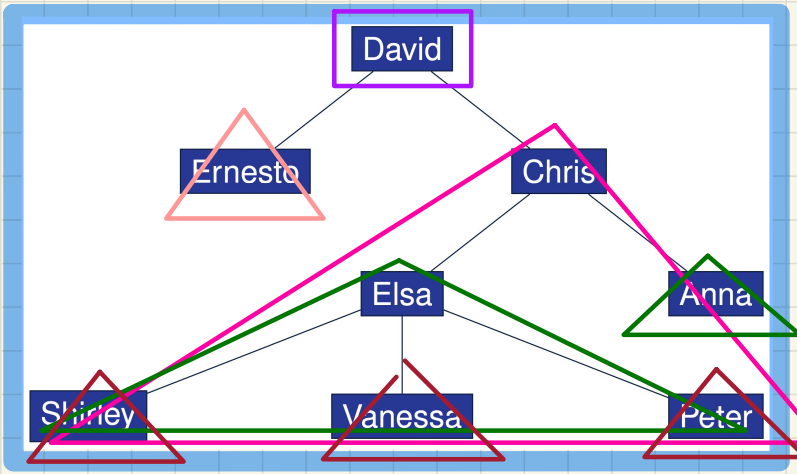
$h = \log(n+1) - 1$

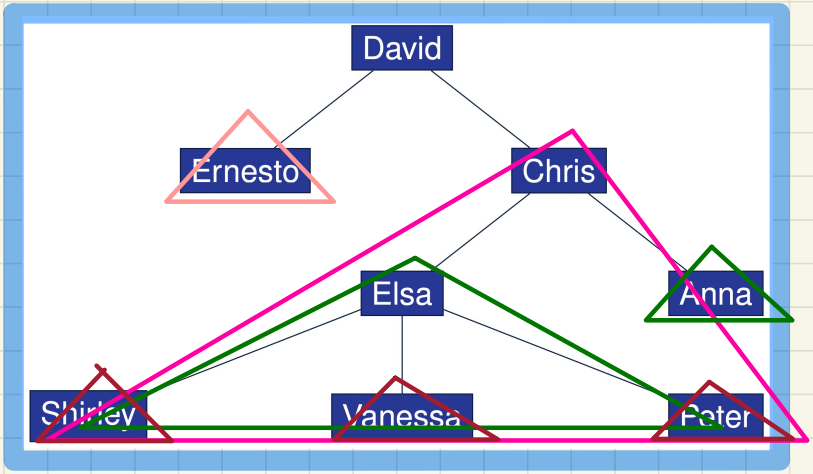
edges from bottom: (max depth)

$7 - 1 = 6$



General Tree Traversals: Pre-Order vs. Post-Order





Pre-Order Traversal from the Root

David Ernesto Chris Elsa S V P Anna

po(E) po(Elsa) po(Anna) po(Chris)

Post-Order Traversal from the Root

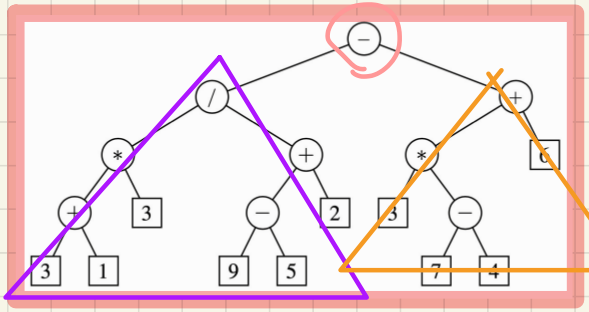
S V P Elsa Anna Chris David

Ernesto po(Elsa) po(1) po(C)

po(E)

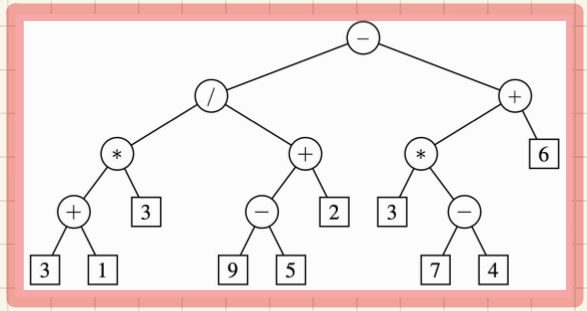
TRAVEL THE WORLD

Binary Tree Traversals

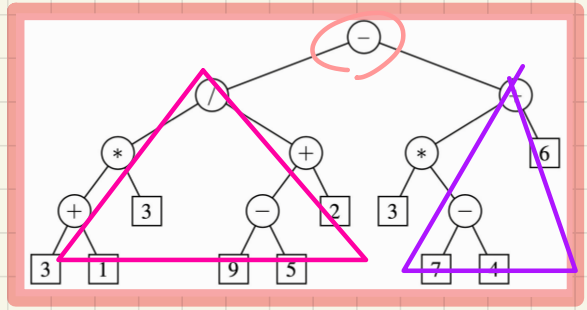


Pre-Order Traversal

− / * + 3 1 3 + − 9 5 2 + * 3 − 7 4 6



In-Order Traversal



Post-Order Traversal

3 1 + 3 * 9 5 − 2 + / 3 7 4 − * 6 + −

Review Q & A - Mar. 10

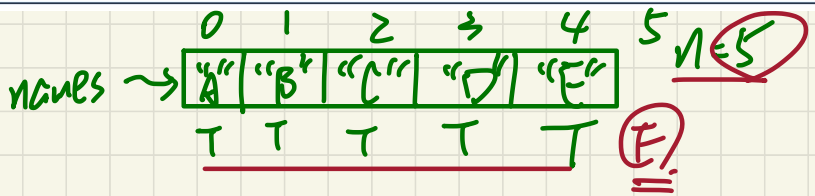
Written Test

Asymptotic Analysis
Instantiating Generics

```
① boolean foundEmptyString = false;
② int i = 0;
③④ while (!foundEmptyString && i < names.length) {
④⑤     if (names[i].length() == 0) {
⑤         /* set flag for early exit */
⑤         foundEmptyString = true;
⑤     }
⑥     i = i + 1;
④ }
```

worst case is when the "" is never found

→ # iterations = names.length = n



n = 5

① 1

② 1

③ (n+1) · 4

④, ⑤, ⑥ 6 · n

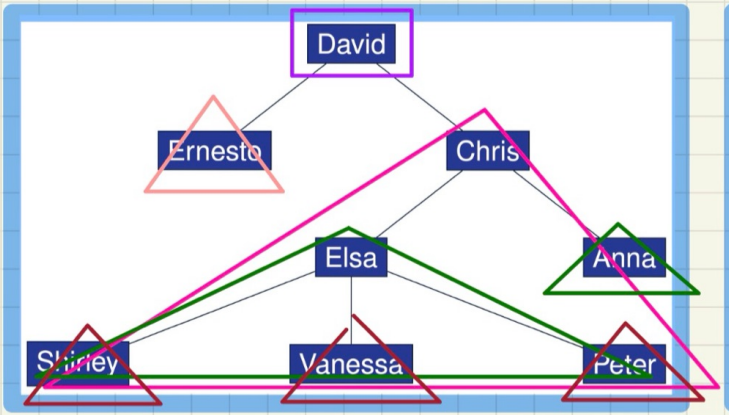
iterations : n

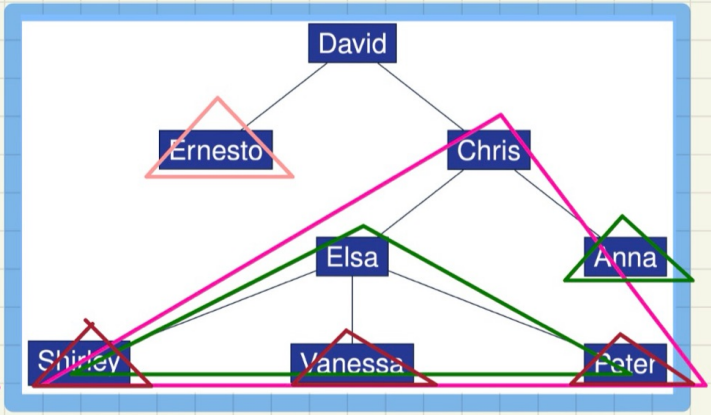
times while cond. is evaluated : n+1 (F)

1 + 1 + (4n+4) + 6n

= 10n + 6

General Tree Traversals: Pre-Order vs. Post-Order





Pre-Order Traversal from the Root

David Ernesto po(E) Chris Elsa S V P po(Elsa) po(Chris) Anna po(Anna)

Post-Order Traversal from the Root

Ernesto po(E) S V P Elsa po(Elsa) Anna po(A) Chris po(C) David

a1.length == n a2.length == m

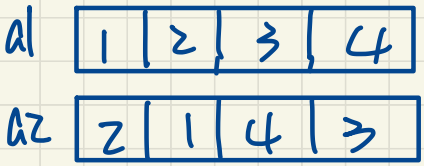
```
boolean contentsMatch(int[] a1, int[] a2){
    for(int i=0; i< a1.length; i++){
        for(int j=0; j< a2.length; j++){
            // check to see if a2[j] == a1[i]   O(1)
        }
    }
    for(int i=0; i< a2.length; i++){
        for(int j=0; j< a1.length; j++){
            // check to see if a1[j] == a2[i]   O(1)
        }
    }
}
```

n · m

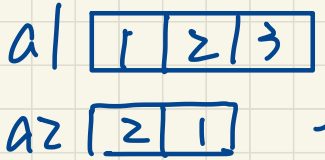
$O(n \cdot m \cdot 1 + m \cdot n \cdot 1)$

$= O(2 \cdot n \cdot m)$

$= O(n \cdot m)$ m · n



true



false

$S1 = S2 \iff S_1 \subseteq S_2 \wedge S_2 \subseteq S_1$

highest power

$5n^2 + 3n \cdot \log n + 2n + 5$ is $O(n^2)$ $[c = 15, n_0 = 1]$

$f(n)$ $g(n)$

Proof

choose $C = |5| + |3| + |2| + |5| = 15$

$n_0 = 1$

Verify: $f(1) \leq C \cdot g(1)$

$5 \cdot 1^2 + 3 \cdot 1 \cdot \log 1 + 2 \cdot 1 + 5$

$= 12$

$\leq 15 \cdot 1^2 = 15$

```
public class MyClass<G, H, I> {
    private G[] a;
    public G m1 (I p1, H p2) {
        /* details of implementation omitted */
        return null;
    }
    public void m2 (H p1,  G p2) {
        /* details of implementation omitted */
    }
}
```

```
public class MyClass<G, H, I> {
    private G[] a;
    public G m1 (I p1, H p2) {
        /* details of implementation omitted */
        return null;
    }
    public void m2 (H p1,  G p2) {
        /* details of implementation omitted */
    }
}
```

Now consider the following declarations from another class (which intends to use the above generic class):

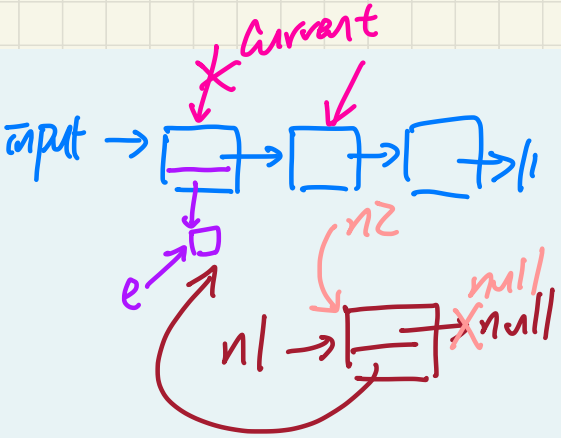
```
Boolean b;
String s;
Integer i;
MyClass<Integer, String, Boolean> obj1;
MyClass<Boolean, String, Integer> obj2;
```

s = obj1.m2 ("alan", 5);	Choose...
b = obj2.m2 ("alan", **false**);	Choose...
obj1.m1 ("alan", 5);	Choose...
obj2.m2 ("alan", **false**);	Choose...
obj1.m2 (5, "alan");	Choose...
obj2.m2 (**false**, "alan");	Choose...
i = obj1.m1 (**true**, "mark");	Choose...
b = obj2.m1 (3, "mark");	Choose...
b = obj1.m1 (**true**, "mark");	Choose...
i = obj2.m1 (3, "mark");	Choose...

Consider the following method which intends to reverse the input chain of nodes:

```
public Node<String> reverseOf(Node<String> input) {
    Node<String> n2 = null;
    Node<String> current = input;
    while(current != null) {
        String e = current.getElement();
        Node<String> n1 = new Node<>(e, null);

        /* missing some line(s) of code */

        current = current.getNext();
    }
    return n2;
}
```

n1.setNext(n2);
n2 = n1;



In the above method, some line or lines of code are missing (from where the comment is) in order for the implementation to be correct. Choose **the** line or **all** lines that are needed.

- [ ] n1.setNext(n2);
- [ ] n2 = n1;
- [ ] n2.setNext(n1);
- [ ] n1.setNext(current);
- [ ] n1 = n2;
- [ ] n2 = current;

Your answer is incorrect.

The correct answers are:

n1.setNext(n2);,

n2 = n1;

Lecture 16 - March 11

Binary Trees, Binary Search

Bounding Internal vs. External Nodes
Proper Binary Trees
Binary Search: Ideas, Java

Announcements/Reminders

- Assignment 3 (on linked Trees) released
- WrittenTest guide & example questions released
- WrittenTest review session materials posted
- Makeup Lecture (on ADTs, Stacks) posted
- Lecture notes template, Office Hours, TA Contact

BT Properties: Bounding # of External Nodes

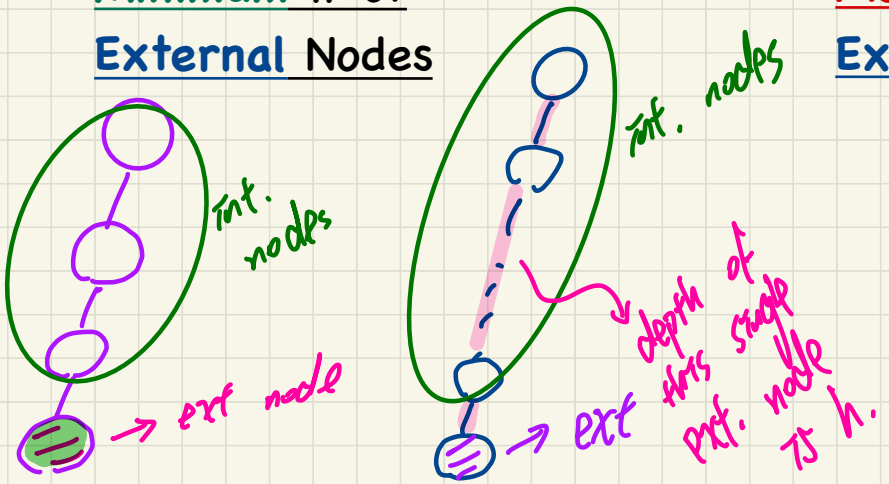
Given a *binary tree* with ***height*** h, the ***number of external nodes*** n_E is bounded as:

$$1 \leq n_E \leq 2^h$$

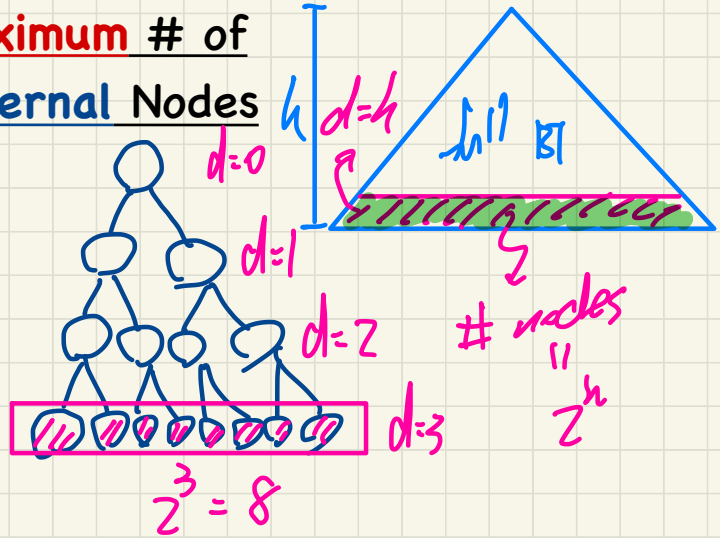
For example, say h = 3

max # edges from bottom node to root

Minimum # of External Nodes



Maximum # of External Nodes



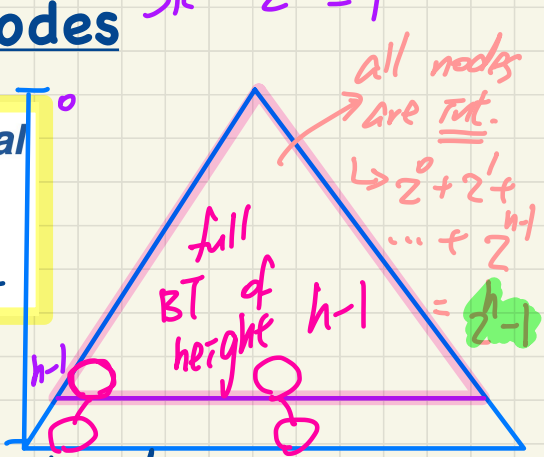
BT Properties: Bounding # of Internal Nodes

$S_k = 2^k - 1$

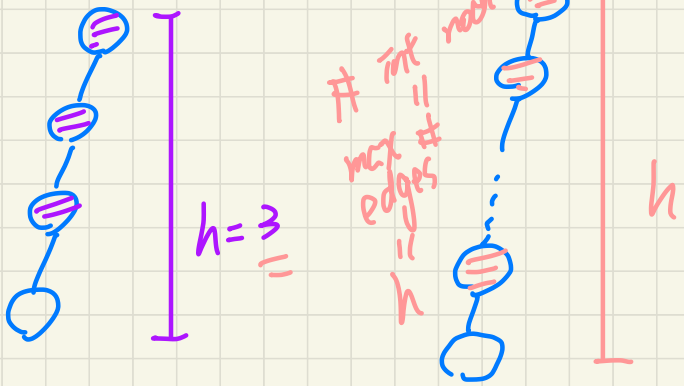
Given a *binary tree* with ***height*** h, the ***number of internal nodes*** n_I is bounded as:

$$h \le n_I \le 2^h - 1$$

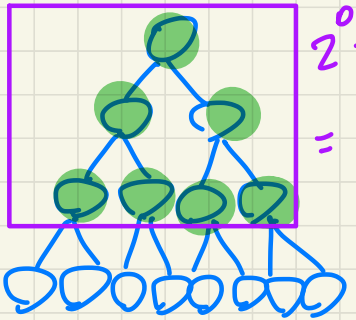
For example, say h = 3



Minimum # of Internal Nodes



Maximum # of Internal Nodes



$2^0 + 2^1 + 2^2$

$= 2^3 - 1$

$= 7$

↳ as long as each node at level h-1 has at least one child, it's ok.

BT Properties: Relating #s of Ext. and Int. Nodes

before the ext. n_I n_E | n_I' n_E' after ext.

Given a *binary tree* that is:

- ***nonempty*** and ***proper***
- with n_I ***internal nodes*** and n_E ***external nodes***

We can then expect that: $n_E = n_I + 1$ ✓✓

Induction on Size of Proper BT

① Base Case: Singleton Proper BT REVIEW!

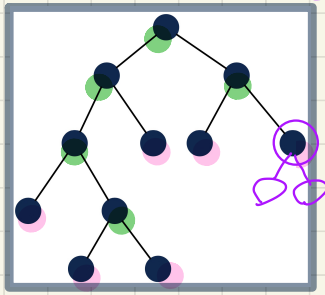
$n_E = 1$

$n_I = 0$

↳ property holds.

② Inductive Hypothesis (I.H.): for a proper BT of size >1: $n_E = n_I + 1$

③ Given a proper binary tree, extend it by choosing the right-most ext. node n and adding two child nodes to it x, y

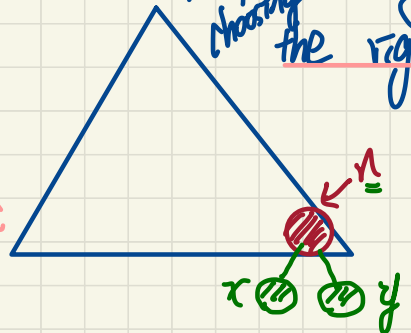


④ $n_E' \overset{(1)}{=} n_E + 1 \overset{IH}{=} (n_I + 1) + 1 \overset{(2)}{=} n_I' + 1$

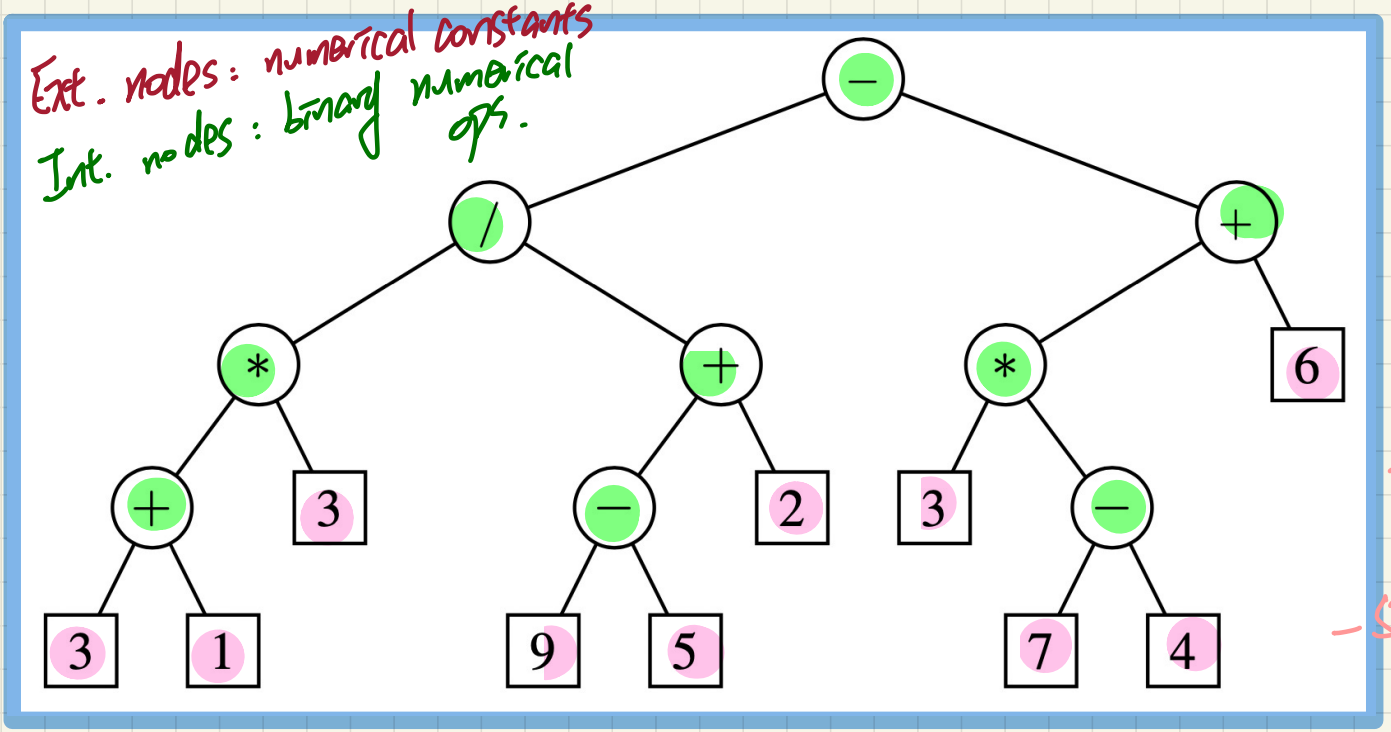
(1) $n_E' = n_E - 1 + 2 = n_E + 1$ (n: -1; x, y: $+2$)

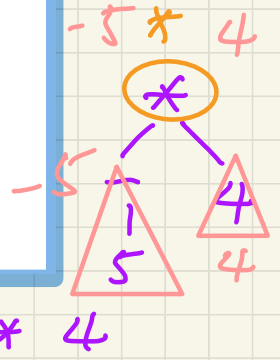
(2) $n_I' = n_I + 1$ (n: $+1$)

Show: $n_E' = n_I' + 1$



Applications of Binary Trees: Infix Notation



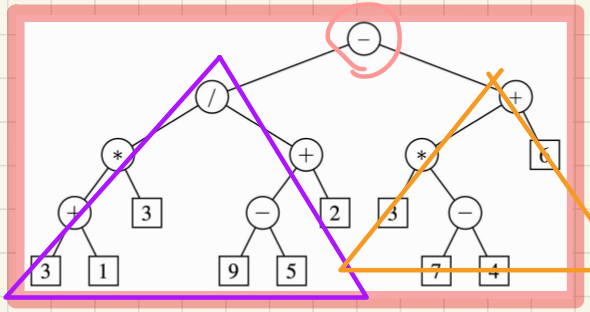


-5 * 4

Q. Is the binary tree necessarily proper?

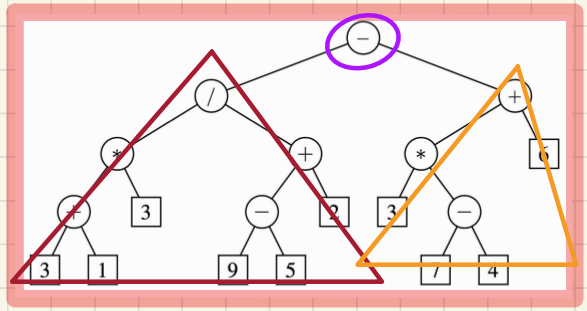
Binary Tree Traversals

TRAVEL THE WORLD



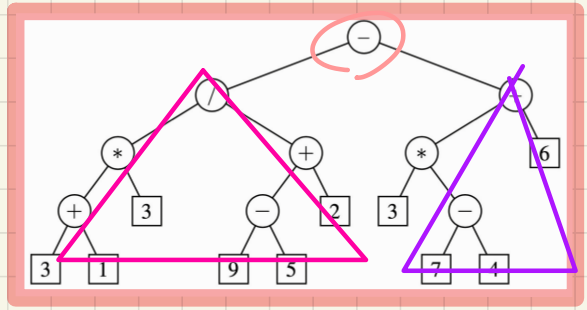
Pre-Order Traversal

− / * + 3 1 3 + − 9 5 2 + * 3 − 7 4 6



In-Order Traversal

3 + 1 * 3 / 9 − 5 + 2 − 3 * 7 − 4 + 6

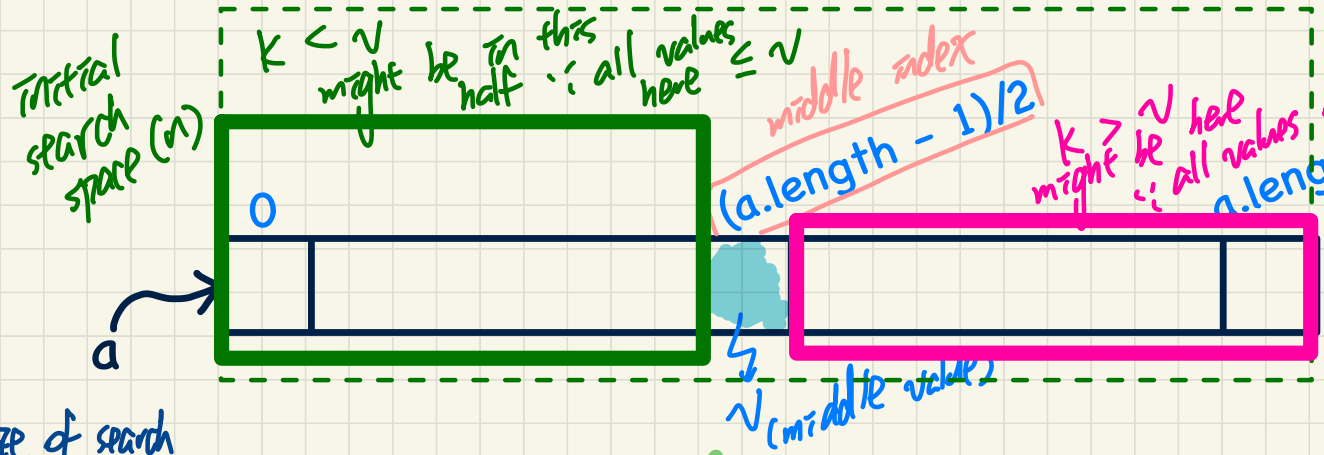


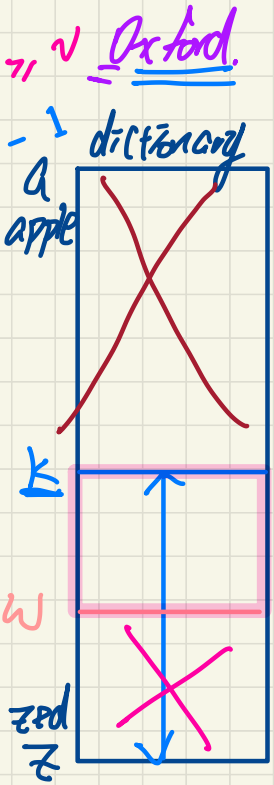
Post-Order Traversal

3 1 + 3 * 9 5 − 2 + / 3 7 4 − * 6 + −

Binary Search: Ideas

Precondition: Array **sorted** in non-descending order

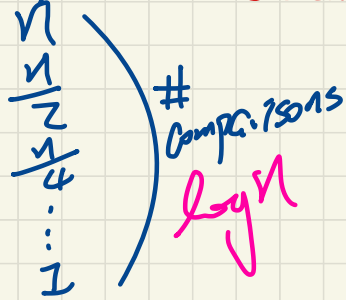




Search: Does key **k** exist in array **a**?

① keep accessing the middle of the search space (halved each time)

② Recurse on: (A) left if $k < v$
(B) right if $k > v$

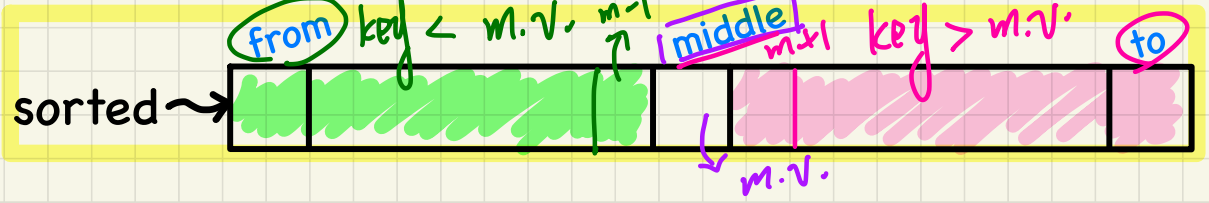


Binary Search in Java

```
boolean binarySearch(int[] sorted, int key) {
  return binarySearchH(sorted, 0, sorted.length - 1, key);
}
boolean binarySearchH(int[] sorted, int from, int to, int key) {
  if (from > to) { /* base case 1: empty range */
    return false; }
  else if(from == to) { /* base case 2: range of one element */
    return sorted[from] == key; }
  else {
    int middle = (from + to) / 2;
    int middleValue = sorted[middle];
    if(key < middleValue) {
      return binarySearchH(sorted, from, middle - 1, key);
    }
    else if (key > middleValue) {
      return binarySearchH(sorted, middle + 1, to, key);
    }
    else { return true; }
  }
}
```

recursive cases

k == middleValue.



Lecture 17 - March 18

Binary Search, Merge Sort

Binary Search: Tracing, Running Time
MergeSort: Ideas, Java, Tracing

Announcements/Reminders

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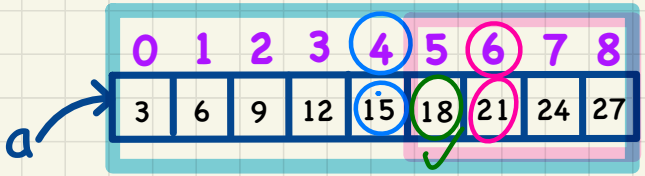
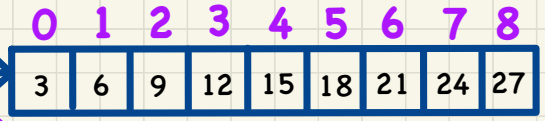
Binary Search: Tracing



initial search space.



Exercise!

search(a,7)

binarySearchH(a,0,8,7)

binarySearchH(a,0,3,7)

binarySearchH(a,2,3,7)

binarySearchH(a,2,1,7)

search(a,18)

key

binarySearchH(a,0,8,18)

$m = \frac{0+8}{2} = 4$

m.v. a[4] = 15

if 7 < 15

↳ go R. ↓ (4+1, 8)

binarySearchH(a,5,8,18)

$m = \frac{5+8}{2} = 6$

m.v. a[6] = 21

if 7 < 21

↳ go L. ↓ (5, 6-1)

binarySearchH(a,5,5,18)

a[5] == 18

true → found it!

Binary Search: Running Time

```
boolean binarySearch(int[] sorted, int key) {
  return binarySearchH(sorted, 0, sorted.length - 1, key);
}
boolean binarySearchH(int[] sorted, int from, int to, int key) {
  if (from > to) { /* base case 1: empty range */
    return false; }
  else if(from == to) { /* base case 2: range of one element */
    return sorted[from] == key; }
  else {
    int middle = (from + to) / 2;
    int middleValue = sorted[middle];
    if(key < middleValue) {
      return binarySearchH(sorted, from, middle - 1, key);
    }
    else if (key > middleValue) {
      return binarySearchH(sorted, middle + 1, to, key);
    }
    else  { return true; }
  }
}
```

n

T(n) = ?

T(0) = 1

T(1) = 1

O(1)

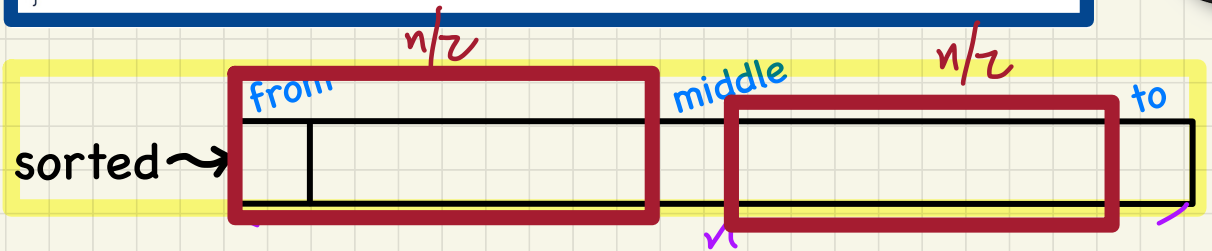
T(n)

n/2

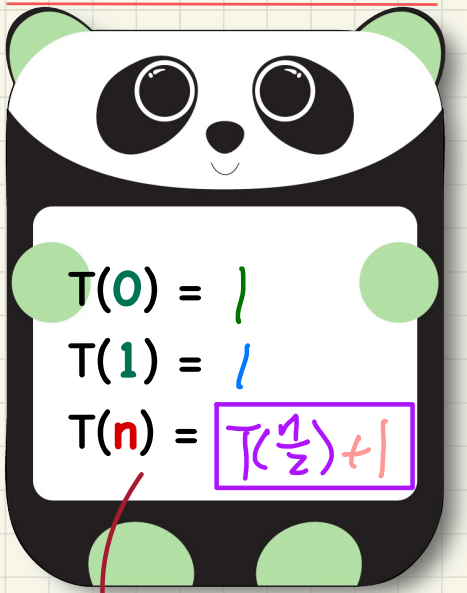
Assume: $T(\frac{n}{2})$

Assume: $T(\frac{n}{2})$

n/2



Running Time as a Recurrence Relation



wrong: (if-else)

$T(n) = T(\frac{n}{2}) + T(\frac{n}{2}) + 1$

Running Time: Unfolding Recurrence Relation

$T(0) = 1$

$T(1) = 1$

$T(n) = T(n/2) + 1$

Assume without loss of generality: $n = 2^x$ for $x \geq 0$

Warm-up: $T(\frac{n}{2}) = T(\frac{n}{2}/2) + 1 = T(\frac{n}{4}) + 1$

$T(n) = T(\frac{n}{2}) + 1$

$= (T(\frac{n}{4}) + 1) + 1$

$= ((T(\frac{n}{8}) + 1) + 1) + 1$

$\vdots$

$1 \quad T(1) + 1 + 1 + \cdots + 1$ (how many? $\log n$ (hm))

$= 1 + \log n \cdot 1 = \log n$

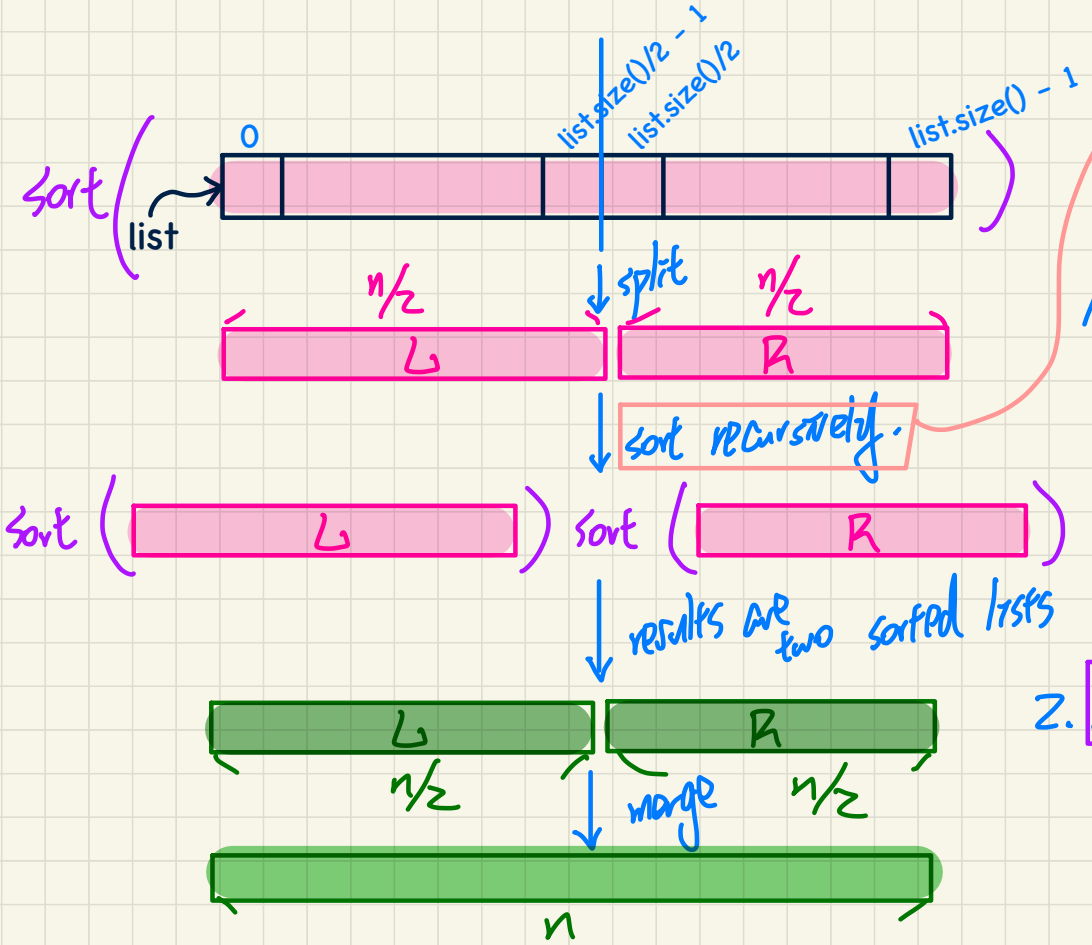
$1 = \frac{n}{n} = \frac{n}{2^{hm}} = \frac{n}{2^{\log n}}$

$n = 8$

$\frac{8}{2^{\log 8}} = 1$

Merge Sort: Ideas

- split
- merge



Each recursive call:

1. split ($O(1)$)
   $n \to \frac{n}{2}, \frac{n}{2}$,
   $\frac{n}{2} \to \frac{n}{4}, \frac{n}{4}$,
   $\frac{n}{4} \to \frac{n}{8}, \frac{n}{8}$,
   ⋮
   $2 \to 1, 1$ (sorted)

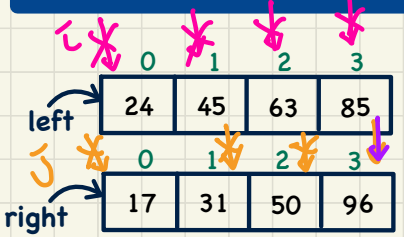
2. Merge ($O(n)$)
   $(1, 1 \to 2,$
   $2, 2 \to 4,$
   ⋮
   $\frac{n}{2}, \frac{n}{2} \to n)$

Merge Sort in Java

```java
public List<Integer> sort(List<Integer> list) {
  List<Integer> sortedList;
  if(list.size() == 0) { sortedList = new ArrayList<>(); }
  else if(list.size() == 1) {
    sortedList = new ArrayList<>();
    sortedList.add(list.get(0));
  }
  else {
    int middle = list.size() / 2;
    List<Integer> left = list.subList(0, middle);
    List<Integer> right = list.subList(middle, list.size());
    List<Integer> sortedLeft = sort(left);
    List<Integer> sortedRight = sort(right);
    sortedList = merge(sortedLeft, sortedRight);
  }
  return sortedList;
}
```

base cases

O(L.size + R.right)



L is exhausted

Precondition

L and R sorted.

Total # iterations for merge: (L.size + j) + (R.size - j)

* Exhaust either L or R.

e.g. L is exhausted. = L.size + R.size

≈ L.size + j iterations so far

** Append remaining ele. from R.

≈ R.size - j iterations

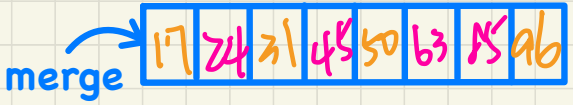
```java
/* Assumption:  L and R are both already sorted. */
private List<Integer> merge(List<Integer> L, List<Integer> R) {
  List<Integer> merge = new ArrayList<>();
  if(L.isEmpty()||R.isEmpty()) { merge.addAll(L); merge.addAll(R); }
  else {
    int i = 0;
    int j = 0;
    while(i < L.size() && j < R.size()) {
      if(L.get(i) <= R.get(j)) { merge.add(L.get(i)); i ++; }
      else { merge.add(R.get(j)); j ++; }
    }
    /* If i >= L.size(), then this for loop is skipped. */
    for(int k = i; k < L.size(); k ++) { merge.add(L.get(k)); }
    /* If j >= R.size(), then this for loop is skipped. */
    for(int k = j; k < R.size(); k ++) { merge.add(R.get(k)); }
  }
  return merge;
}
```

exit: i >= L.size || j >= R.size

O(1)

O(1)

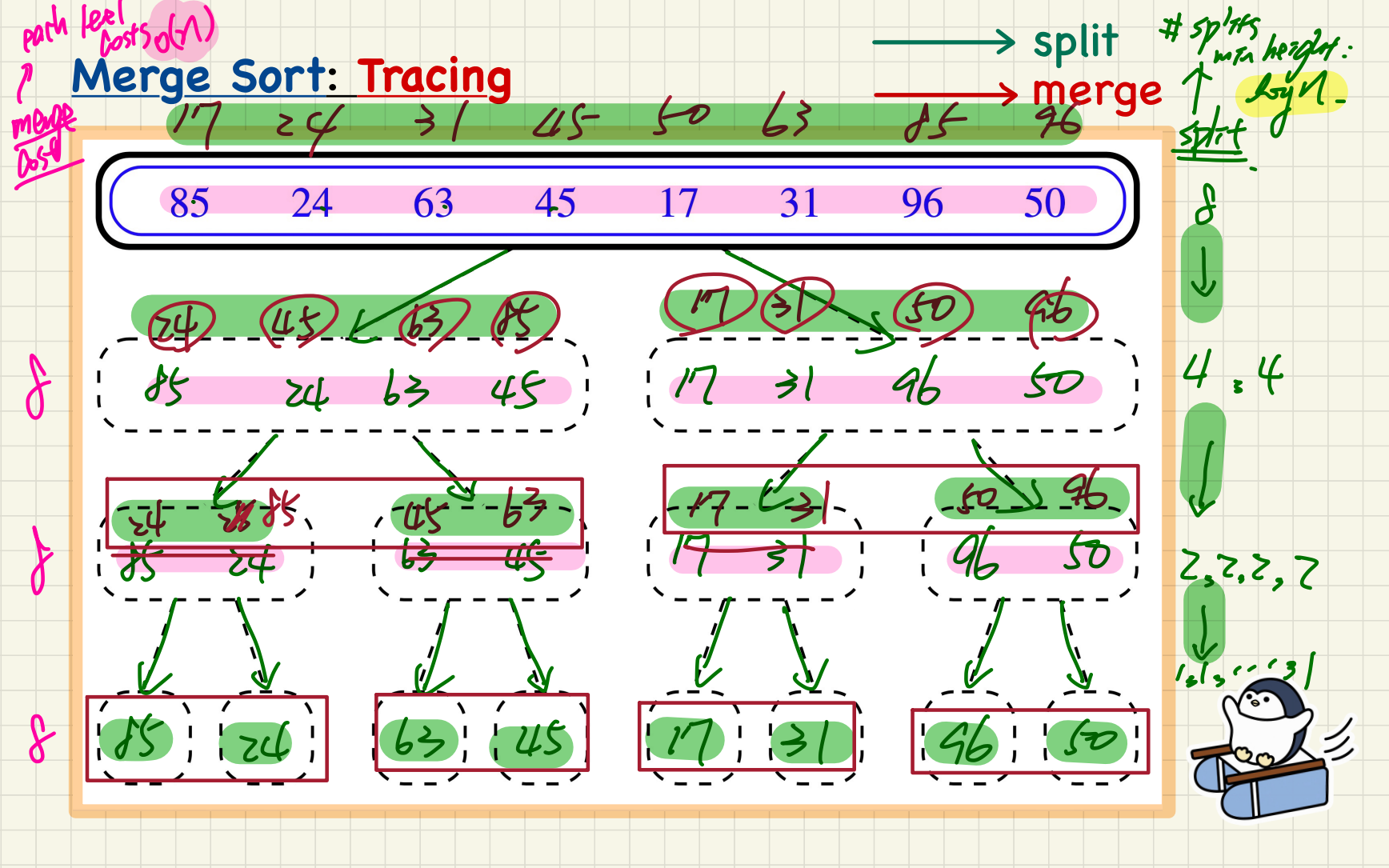
append the remaining elements from the unexhausted list.



Merge Sort: Tracing

split

merge



Lecture 18 - March 20

Merge Sort, Quick Sort, BST

MergeSort: Recurrence Relation
QuickSort: Ideas, Java, RT

Announcements/Reminders

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Merge Sort: Running Time

```
public List<Integer> sort(List<Integer> list) {
  List<Integer> sortedList;
  if(list.size() == 0) { sortedList = new ArrayList<>(); }
  else if(list.size() == 1) {
    sortedList = new ArrayList<>();
    sortedList.add(list.get(0));
  }
  else {
    int middle = list.size() / 2;
    List<Integer> left = list.subList(0, middle);
    List<Integer> right = list.subList(middle, list.size());
    List<Integer> sortedLeft = sort(left);
    List<Integer> sortedRight = sort(right);
    sortedList = merge(sortedLeft, sortedRight);
  }
  return sortedList;
}
```

T(0) = 1

T(1) = 1

O(1)

* RT: size(sortedL) + size(sortedR) = O(n)

1 + n + 2·T(n/2)

O(n)

Assume: T(n/2)

Assume: T(n/2)

list

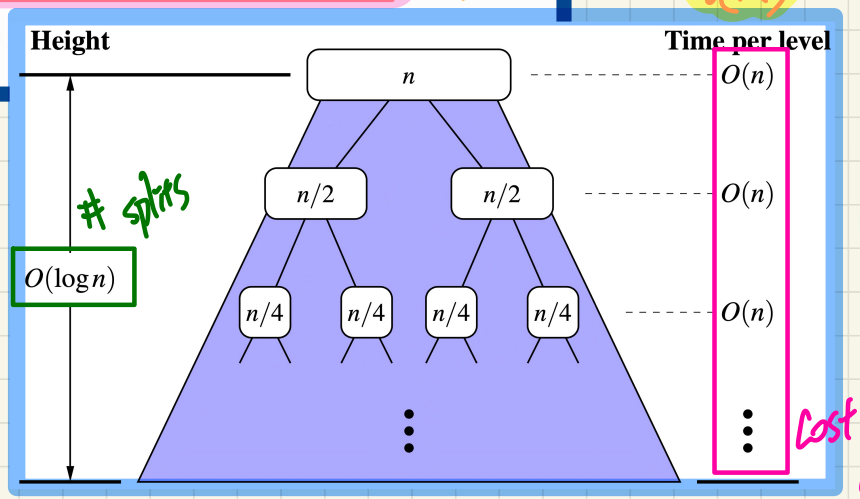
n

L

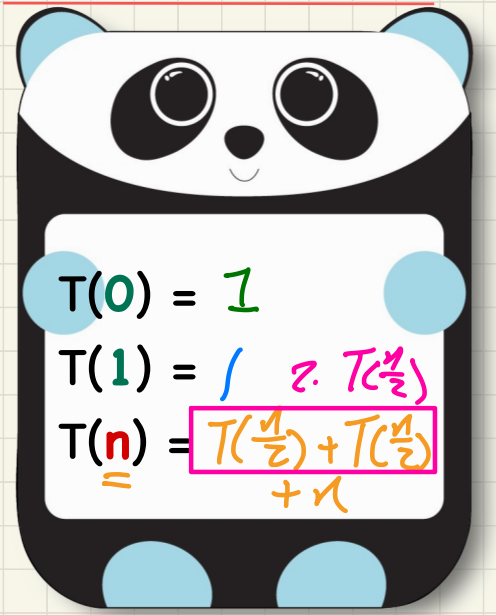
R

n/2

n/2



Running Time as a Recurrence Relation



Running Time: Unfolding Recurrence Relation

$T(0) = 1$

$T(1) = 1$

$T(n) = 2 \cdot T(n/2) + n$

Warm-up: $T(\frac{n}{2}) = 2 \cdot T(\frac{n/2}{2}) + \frac{n}{2}$ ($n/4$)

$T(n) = 2 \cdot T(\frac{n}{2}) + n$

$= 2 \cdot (2 \cdot T(\frac{n}{4}) + \frac{n}{2}) + n$ $\quad [4 \cdot T(\frac{n}{4}) + 2n]$

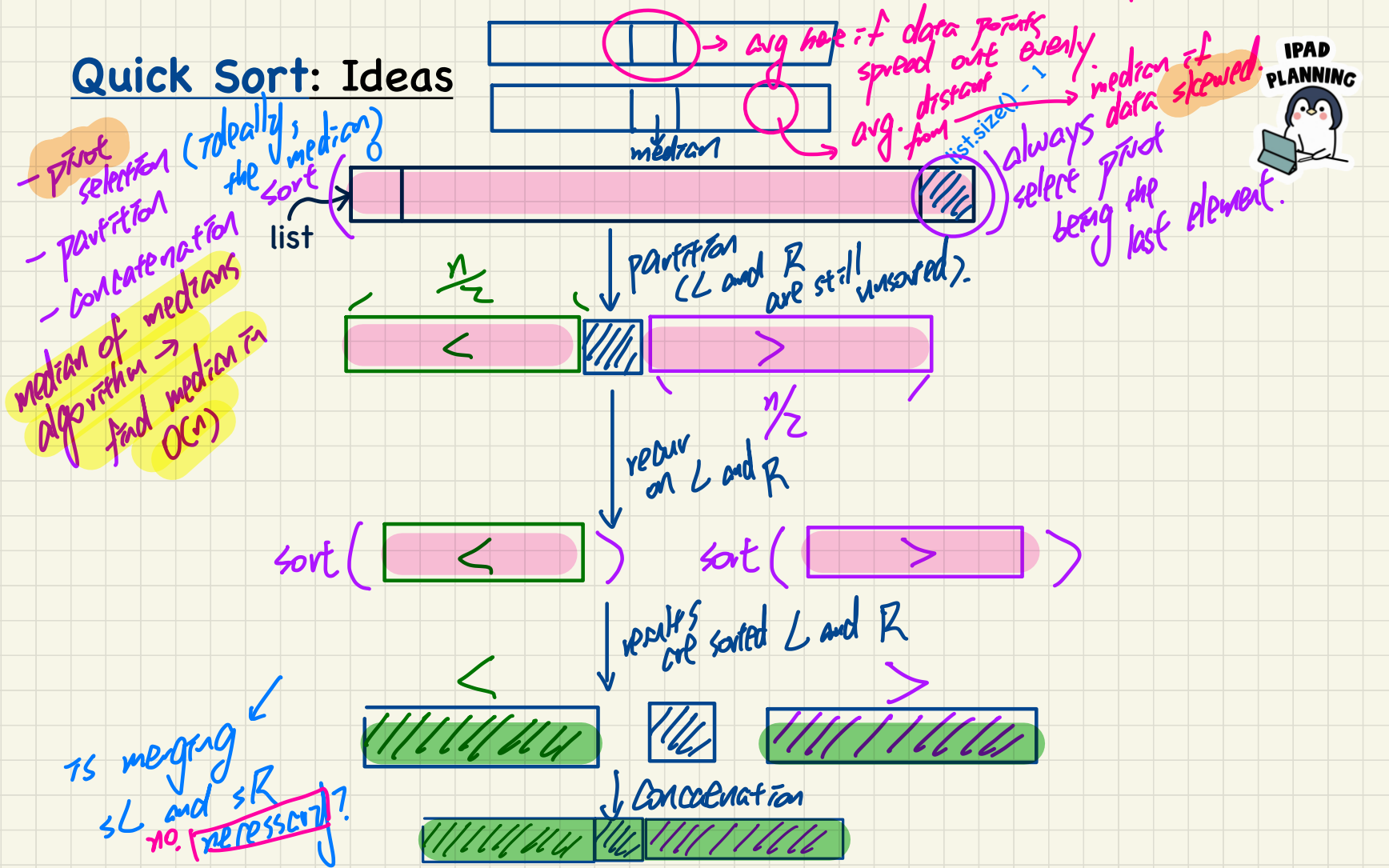
$= 2 \cdot (2 \cdot (2 \cdot T(\frac{n}{8}) + \frac{n}{4}) + \frac{n}{2}) + n$ $\quad [8 \cdot T(\frac{n}{8}) + 3n]$

$\vdots$

$2^? \cdot 1 \; T(1) + ? \cdot n = 2^{\log n} \cdot 1 + \log n \cdot n = n + n \log n = O(n \log n)$

$1 = \frac{n}{n} = \frac{n}{2^{\log n}}$

Quick Sort: Ideas



Quick Sort in Java

Median of medians algo. : O(n)
↳ as opposed to sort and select.

T(n)

```
public List<Integer> sort(List<Integer> list) {
  List<Integer> sortedList;
  if(list.size() == 0) { sortedList = new ArrayList<>(); }
  else if(list.size() == 1) {
    sortedList = new ArrayList<>(); sortedList.add(list.get(0)); }
  else {
    int pivotIndex = list.size() - 1;
    int pivotValue = list.get(pivotIndex);
    List<Integer> left = allLessThanOrEqualTo(pivotIndex, list);
    List<Integer> right = allLargerThan(pivotIndex, list);
    List<Integer> sortedLeft = sort(left);
    List<Integer> sortedRight = sort(right);
    sortedList = new ArrayList<>();
    sortedList.addAll(sortedLeft);
    sortedList.add(pivotValue);
    sortedList.addAll(sortedRight);
  }
  return sortedList;
}
```

O(1)

ideally, this is close to the index of median

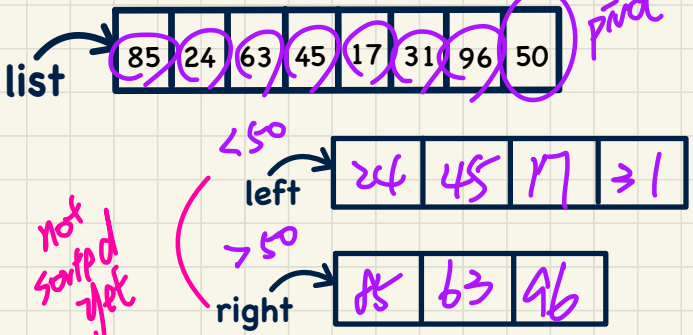
O(n)

T(n/2)

T(n/2)

→ only if pivot ≈ median

const. O(1)




```
List<Integer> allLessThanOrEqualTo(int pivotIndex, List<Integer> list)
  List<Integer> sublist = new ArrayList<>();
  int pivotValue = list.get(pivotIndex);
  for(int i = 0; i < list.size(); i ++) {
    int v = list.get(i);
    if(i != pivotIndex && v <= pivotValue) { sublist.add(v); }
  }
  return sublist;
}
List<Integer> allLargerThan(int pivotIndex, List<Integer> list) {
  List<Integer> sublist = new ArrayList<>();
  int pivotValue = list.get(pivotIndex);
  for(int i = 0; i < list.size(); i ++) {
    int v = list.get(i);
    if(i != pivotIndex && v > pivotValue) { sublist.add(v); }
  }
  return sublist;
}
```

Quick Sort: Tracing

best case for # partitions: log

→ split

→ concatenate

17 24 31 45 50 63 85 96

85 24 63 45 17 31 96 **50**

pivot partition

17 24 31 45 < 50

63 85 96 > 50

24 45 17 **31** 50 85 63 **96**

<63 < 85 > + 96 f <>

17 24 31 45 <17, 24> + 31 + <85>

63 85 96

24 17 31 45 85 63 96

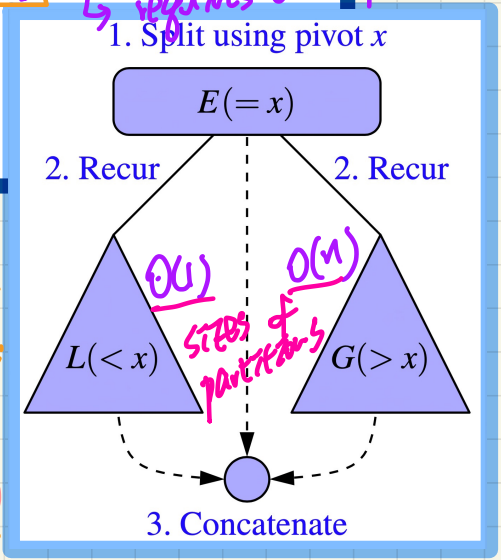
<> + 17 + <24>

<> + 63 + <85>

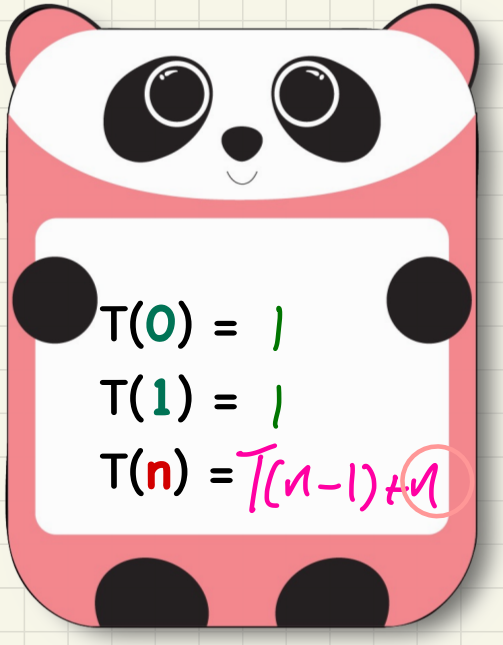
17 24 63 85

Quick Sort: Worst-Case Running Time

```
public List<Integer> sort(List<Integer> list) {
  List<Integer> sortedList;
  if(list.size() == 0) { sortedList = new ArrayList<>(); }
  else if(list.size() == 1) {
    sortedList = new ArrayList<>(); sortedList.add(list.get(0)); }
  else {
    int pivotIndex = list.size() - 1;
    int pivotValue = list.get(pivotIndex);
    List<Integer> left = allLessThanOrEqualTo(pivotIndex, list);
    List<Integer> right = allLargerThan(pivotIndex, list);
    List<Integer> sortedLeft = sort(left);
    List<Integer> sortedRight = sort(right);
    sortedList = new ArrayList<>();
    sortedList.addAll(sortedLeft);
    sortedList.add(pivotValue);
    sortedList.addAll(sortedRight);
  }
  return sortedList;
}
```



Running Time as a Recurrence Relation

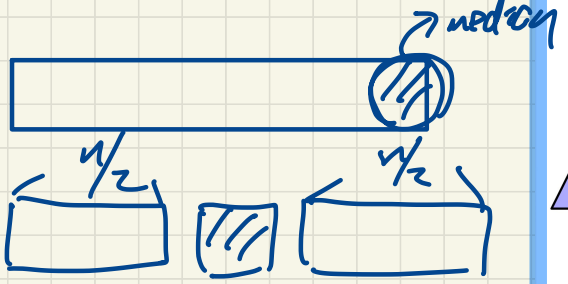


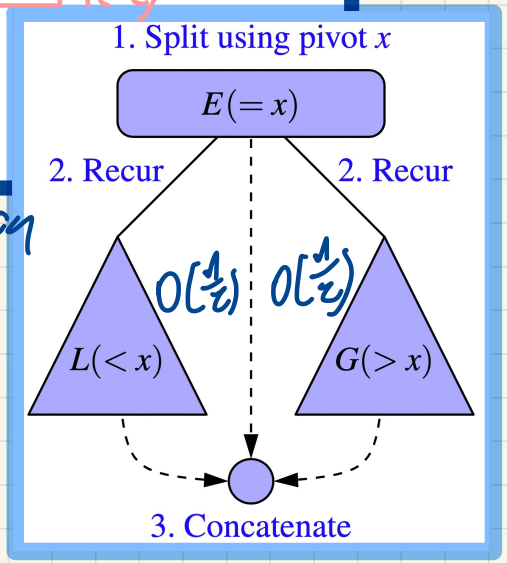
Quick Sort: Best-Case Running Time

```
public List<Integer> sort(List<Integer> list) {
  List<Integer> sortedList;
  if(list.size() == 0) { sortedList = new ArrayList<>(); }
  else if(list.size() == 1) {
    sortedList = new ArrayList<>(); sortedList.add(list.get(0)); }
  else {
    int pivotIndex = list.size() - 1;
    int pivotValue = list.get(pivotIndex);
    List<Integer> left = allLessThanOrEqualTo(pivotIndex, list);
    List<Integer> right = allLargerThan(pivotIndex, list);
    List<Integer> sortedLeft = sort(left);
    List<Integer> sortedRight = sort(right);
    sortedList = new ArrayList<>();
    sortedList.addAll(sortedLeft);
    sortedList.add(pivotValue);
    sortedList.addAll(sortedRight);
  }
  return sortedList;
}
```

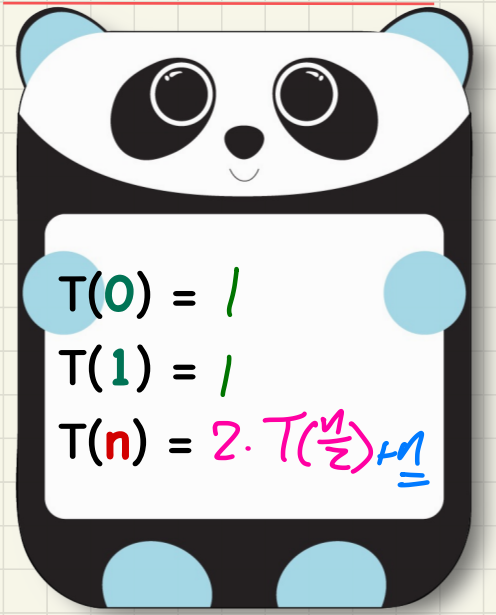
O(n)

T(n/2) T(n/2)





Running Time as a Recurrence Relation



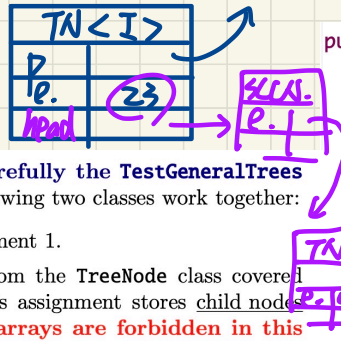
O(n log n)
≈ merge sort

Review Q & A - Mar. 21

Programming Test 2

Assignment 3 Solution

Generic Classes



This assignment requires the more intermediate use of generics. **Study carefully the TestGeneralTrees class given to you (in the tests package)**, which contains how the following two classes work together:

- SLLNode<E>: this class is essentially the Node class you used in Assignment 1.
- TreeNode<E>: this class is similar to but should be distinguished from the TreeNode class covered in Lecture W8. The version of TreeNode class you are given for this assignment stores child nodes as **a chain of singly-linked nodes**. **Remember that primitive arrays are forbidden in this assignment.**
- In the TestGeneralTrees class, pay attention to the following type declarations:
  - `TreeNode<String> n;` declares a tree node storing some string value: we write `n.getElement()` to retrieve the string value. In this case, the generic parameter E declared in the TreeNode class (i.e., TreeNode<E>) is instantiated by String.
  - `TreeNode<Integer> n;` declares a tree node storing some integer value: we write `n.getElement()` to retrieve the integer value. In this case, the generic parameter E declared in the TreeNode class (i.e., TreeNode<E>) is instantiated by Integer.
  - `SLLNode<TreeNode<String>> tn;` declares a singly-linked node storing the reference of some tree node (which in turn stores some string value): we write `tn.getElement()` to retrieve the tree node (of type TreeNode<String>) and write `tn.getElement().getElement()` to retrieve the stored string value. In this case, the generic parameter E declared in the SLLNode class (i.e., SLLNode<E>) is instantiated by TreeNode<String> (which in tern instantiates the generic parameter E in the TreeNode class by String).
  - `SLLNode<TreeNode<Integer>> tn;` declares a singly-linked node storing the reference of some tree node (which in turn stores some integer value): we write `tn.getElement()` to retrieve the tree node (of type TreeNode<Integer>) and write `tn.getElement().getElement()` to retrieve the stored integer value. In this case, the generic parameter E declared in the SLLNode class (i.e., SLLNode<E>) is instantiated by TreeNode<Integer> (which in tern instantiates the generic parameter E in the TreeNode class by Integer).

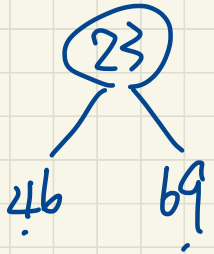
```
public class TreeNode<E> {
    private E element; /* data object */
    private TreeNode<E> parent; /* unique parent
    private SLLNode<TreeNode<E>> headOfChildList
    private SLLNode<TreeNode<E>> tailOfChildList

    public TreeNode(E element) {

    public E getElement() {

    public void setElement(E element) {

    public TreeNode<E> getParent() {

    public void setParent(TreeNode<E> parent) {

    public SLLNode<TreeNode<E>> getChildren() {

    public void addChild(TreeNode<E> child) {
}
```

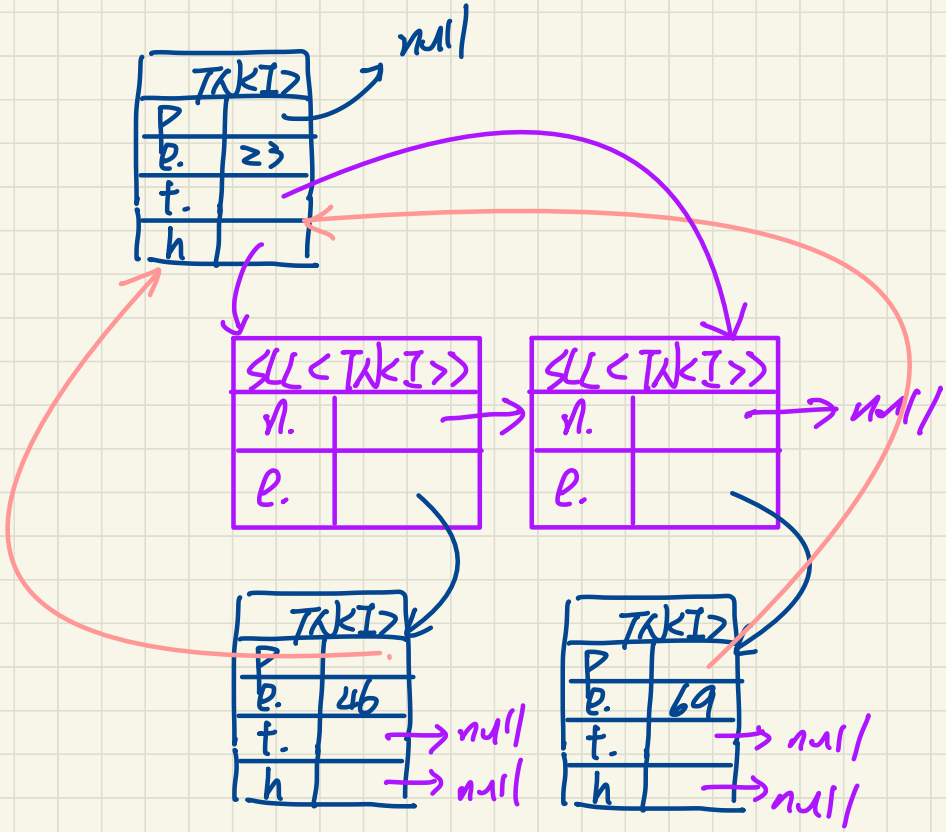
head.getE. → TN<E>

head.getE().getE()

TN<E>

```
public class SLLNode<E> {
    private E element;
    private SLLNode<E> next;

    public SLLNode(E e, SLLNode<E> n) {

    public E getElement() {

    public SLLNode<E> getNext() {

    public void setNext(SLLNode<E> n) {

    public void setElement(E e) {
}
```





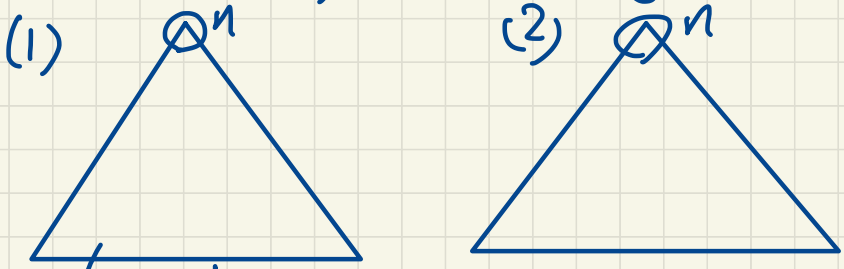
A3 starter

↳ tests

↳ TestsTrees.java

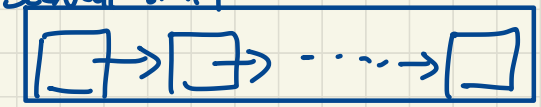
Task 1 Rank

task1(TN n, int i, int j)



(a) pre-order or post-order traversal.

SLNode chain



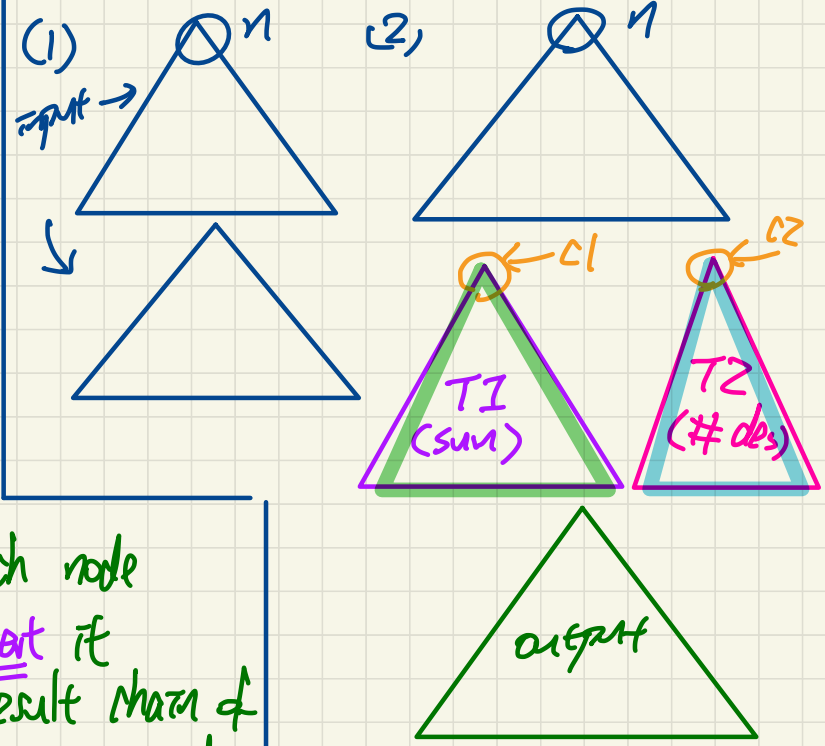
(a) traverse, and when each node is added, insert it s.t. the result chain of nodes remains sorted.

(b) Sort the traversal result

↳ e.g. insertion sort or selection sort

(c) extract nodes from indices i to j.

Task 2: Stats (sum of values, # des.)



Lecture 19 - March 25

Binary Search Trees

BST: Search Property
BST: Sorting Property
BST: Constructing BST Nodes

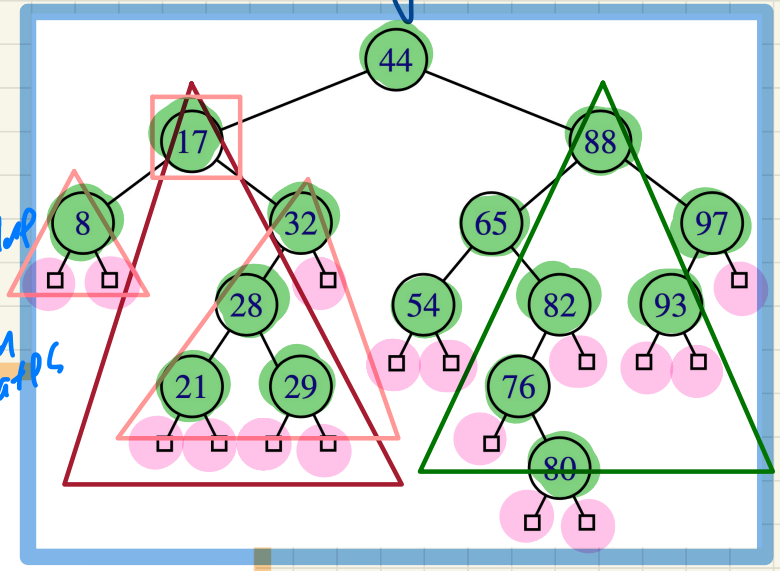
Announcements/Reminders

- Assignment 4 (on linked Trees) to be released
- ProgTest2 info & example questions released
- Assignment 3 (on linked Trees) solution released
- WrittenTest and ProgTest1 results & feedback released
- Makeup Lecture (on Queues) posted
- Lecture notes template, Office Hours, TA Contact

Binary Search Trees: Recursive Definition

Search Property should hold recursively on the root of every subtree.

- external node
- internal node
  - + LST
  - + RST



for searching ⇒ no duplicates

data value ⇒ may contain duplicates

Node p stores
(key(p), value(p))

p

LST <

$\forall n.\ n \in LST(p) \Rightarrow key(n) < key(p)$

Each node n of LST is such that key(n) < key(p)

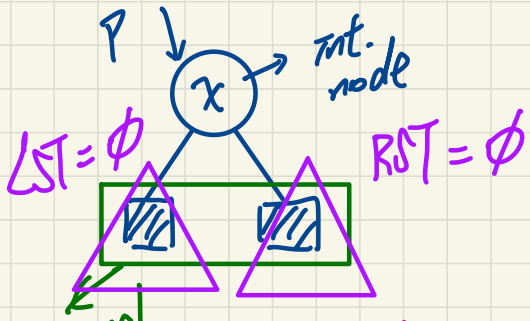
>

RST

Each node n of RST is such that key(n) > key(p)

$\forall n : n \in RST(p) \Rightarrow key(n) > key(p)$

Is a Singleton BT a BST?



(≈ header/trailer in DLN
↳ code structure
↳ does not improve performance)

$$\forall x \cdot \boxed{false \Rightarrow P(x)}$$

false

$$\forall n: \boxed{n \in \cancel{LST(p)}^{\emptyset}} \Rightarrow key(n) < key(p)$$

zero of $\Rightarrow$: True

$$\wedge$$

false

$$\forall n: \boxed{n \in \cancel{RST(p)}^{\emptyset}} \Rightarrow key(n) > key(p)$$

$$\equiv \boxed{True}$$

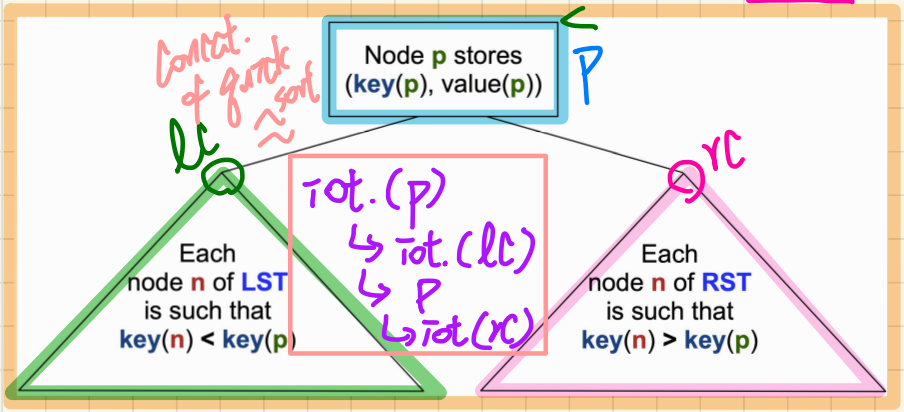
Binary Search Trees: Sorting Property

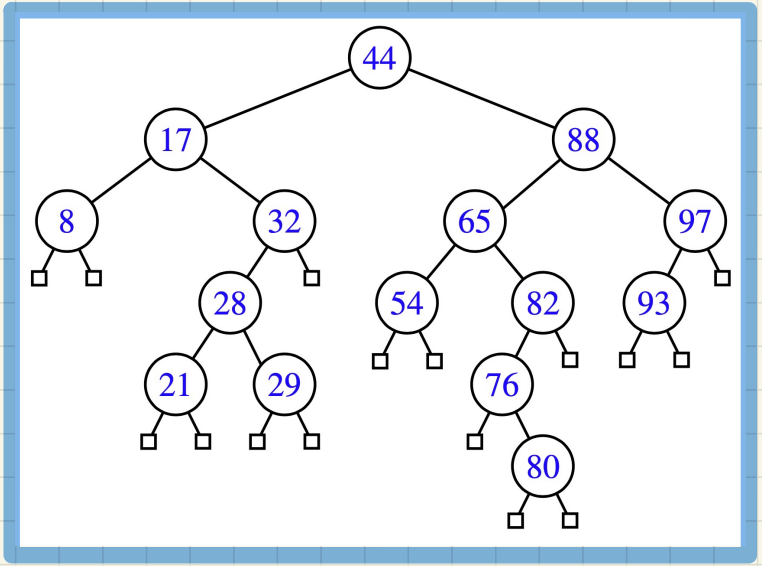
- BST: Non-Linear Structure
- In-Order Traversal

Given a T. that's a BST:

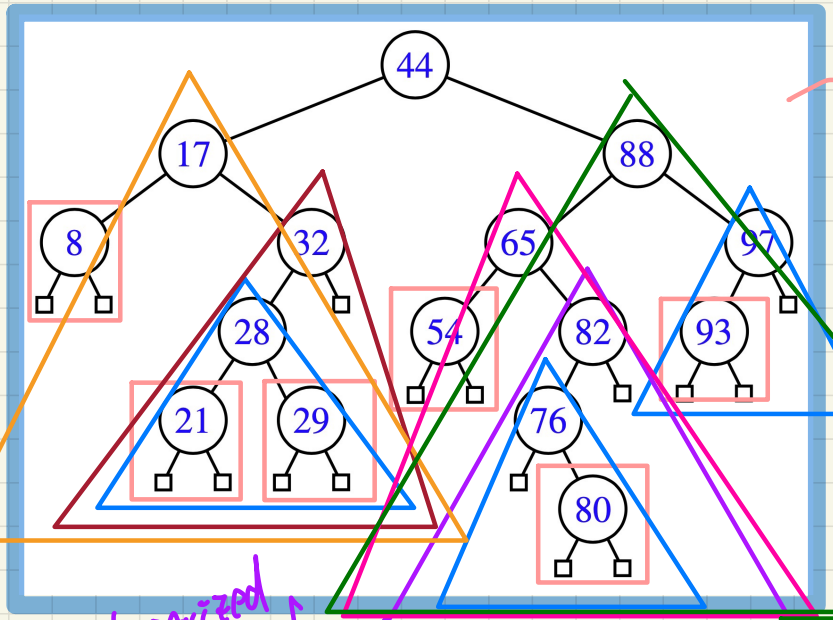
(1) search property sat. (recursively)

(2) in-order traversal: LST, root, RST





Building Sorted Seq. from In-Order Traversal on BST



→ non-linear, hierarchical structure

≈ Concat in quick sort!

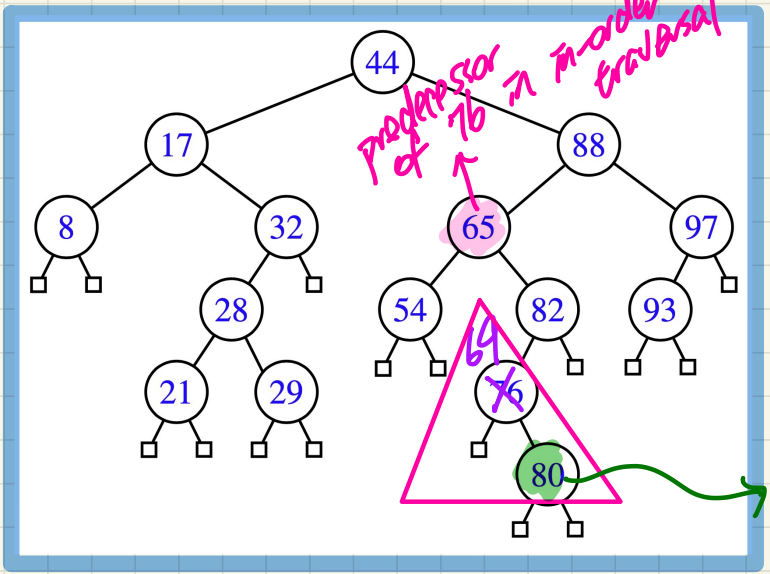
I.O.T: linearized version of the tree

8 17 21 28 29 32 44 54 65 76 80 82 88 93 97

Exercise: Checking the Search Property (1)

Remember: For a BT to be a BST, the Search Property should hold recursively on the root of each subtree.

In-Order: <8, 17, 21, 28, 29, 32, 44, 54, 65, ~~76~~, 80, 82, 88, 93, 97>



69

↓

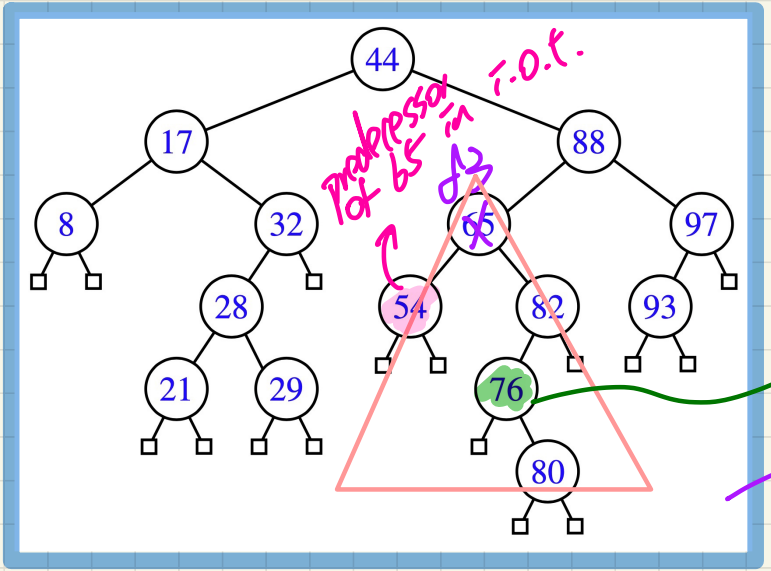
ok :)

65 < 69 < 80

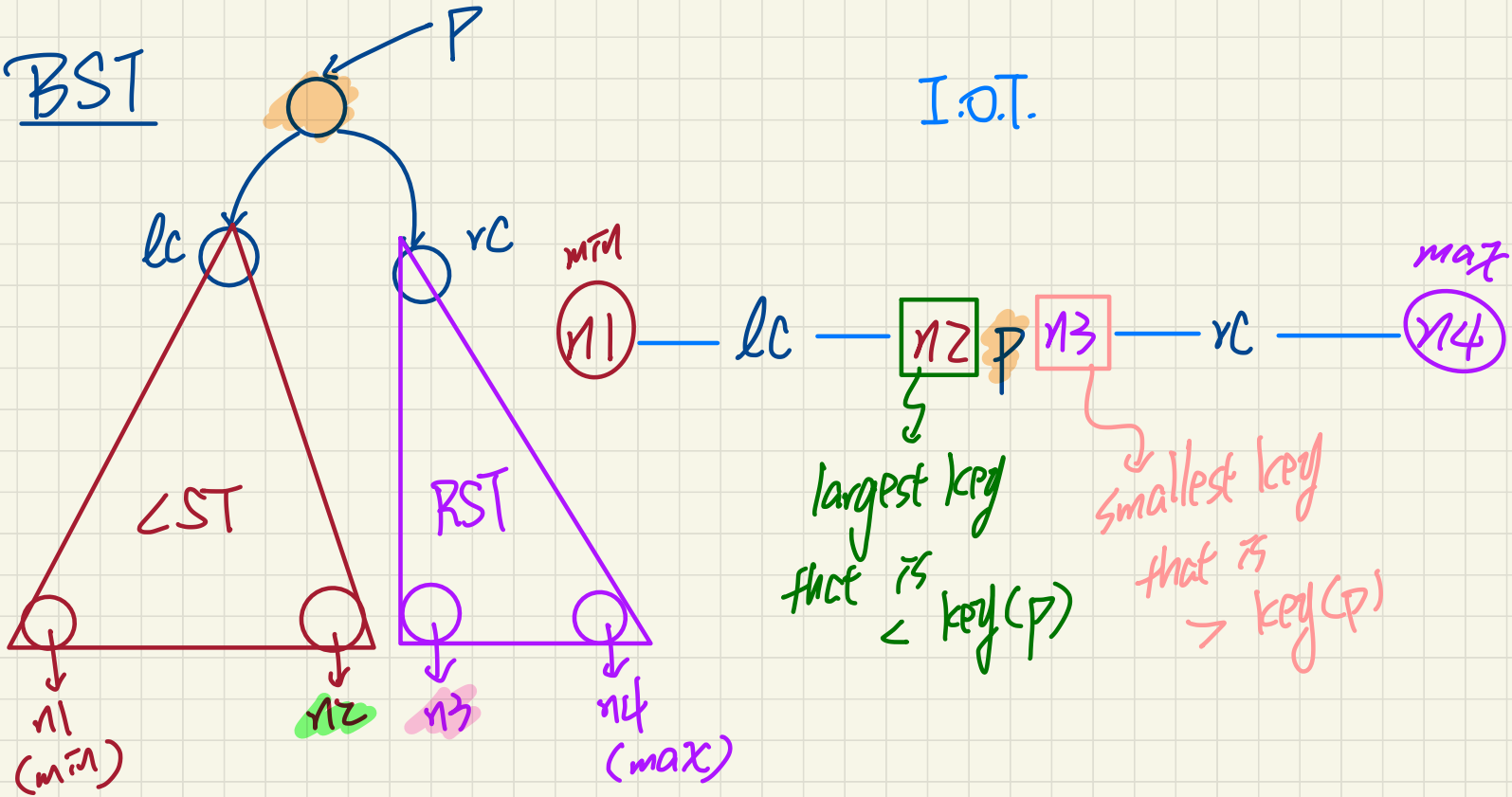
Exercise: Checking the Search Property (2)

Remember: For a BT to be a BST, the Search Property should hold recursively on the root of each subtree.

In-Order: <8, 17, 21, 28, 29, 32, 44, 54, ~~65~~ 63, 76, 80, 82, 88, 93, 97>

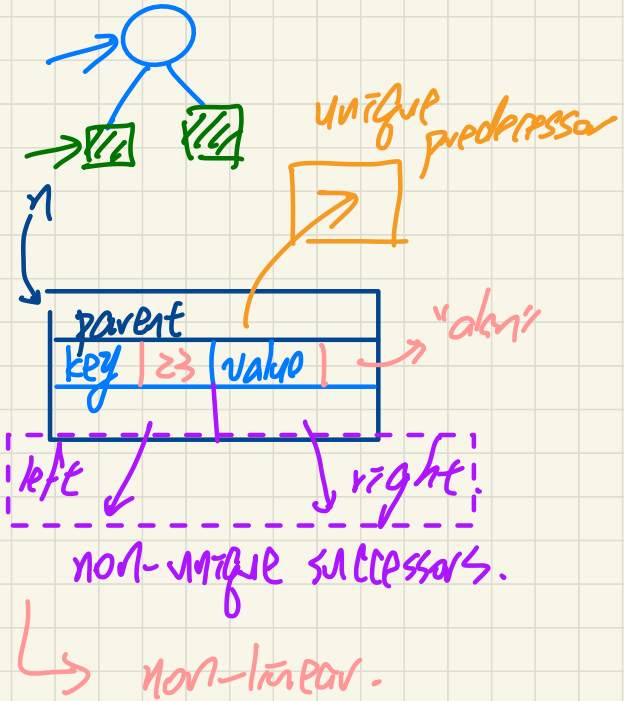


Visual Summary: In-Order Traversal on BST



Generic, Binary Tree Nodes

```
public class BSTNode<E> {
  private int key;          /* key */
  private E value;          /* value */
  private BSTNode<E> parent; /* unique parent node */
  private BSTNode<E> left;  /* left child node */
  private BSTNode<E> right; /* right child node */

  public BSTNode() { ... }
  public BSTNode(int key, E value) { ... }

  public boolean isExternal() {
    return this.getLeft() == null && this.getRight() == null;
  }
  public boolean isInternal() {
    return !this.isExternal();
  }
  public int getKey() { ... }
  public void setKey(int key) { ... }
  public E getValue() { ... }
  public void setValue(E value) { ... }
  public BSTNode<E> getParent() { ... }
  public void setParent(BSTNode<E> parent) { ... }
  public BSTNode<E> getLeft() { ... }
  public void setLeft(BSTNode<E> left) { ... }
  public BSTNode<E> getRight() { ... }
  public void setRight(BSTNode<E> right) { ... }
}
```



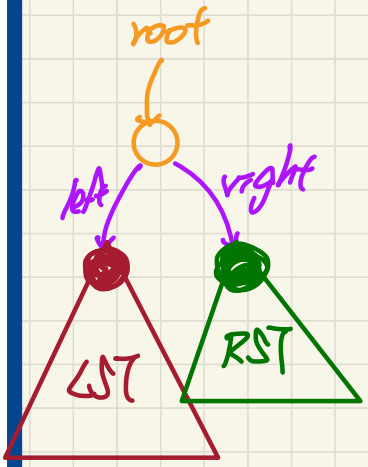
Compare:

+ prev ref.
+ next ref.

in a DLN.

Generic, Binary Tree Nodes - Traversal

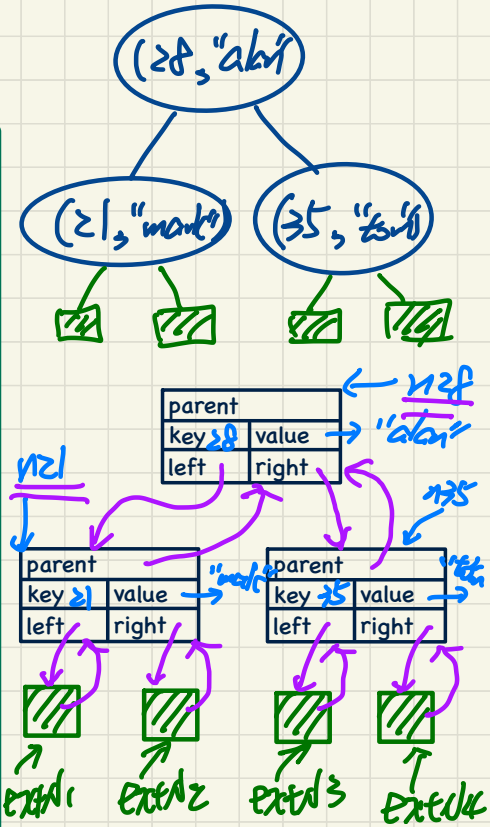
```
import java.util.ArrayList;
public class BSTUtilities<E> {
  public ArrayList<BSTNode<E>> inOrderTraversal(BSTNode<E> root) {
    ArrayList<BSTNode<E>> result = null;
    if(root.isInternal()) {
      result = new ArrayList<>();

      if(root.getLeft().isInternal) {
        result.addAll(inOrderTraversal(root.getLeft()));
      }

      result.add(root);

      if(root.getRight().isInternal) {
        result.addAll(inOrderTraversal(root.getRight()));
      }
    }
    return result;
  }
}
```



Tracing: Constructing and Traversing a BST

```
@Test
public void test_binary_search_trees_construction() {
  BSTNode<String> n28 = new BSTNode<>(28, "alan");
  BSTNode<String> n21 = new BSTNode<>(21, "mark");
  BSTNode<String> n35 = new BSTNode<>(35, "tom");
  BSTNode<String> extN1 = new BSTNode<>();
  BSTNode<String> extN2 = new BSTNode<>();
  BSTNode<String> extN3 = new BSTNode<>();
  BSTNode<String> extN4 = new BSTNode<>();
  n28.setLeft(n21); n21.setParent(n28);
  n28.setRight(n35); n35.setParent(n28);
  n21.setLeft(extN1); extN1.setParent(n21);
  n21.setRight(extN2); extN2.setParent(n21);
  n35.setLeft(extN3); extN3.setParent(n35);
  n35.setRight(extN4); extN4.setParent(n35);
  BSTUtilities<String> u = new BSTUtilities<>();
  ArrayList<BSTNode<String>> inOrderList = u.inOrderTraversal(n28);
  assertTrue(inOrderList.size() == 3);
  assertEquals(21, inOrderList.get(0).getKey());
  assertEquals("mark", inOrderList.get(0).getValue());
  assertEquals(28, inOrderList.get(1).getKey());
  assertEquals("alan", inOrderList.get(1).getValue());
  assertEquals(35, inOrderList.get(2).getKey());
  assertEquals("tom", inOrderList.get(2).getValue());
}
```

sorted in-order seq.



parent	
key	value
left	right

Lecture 20 - April 1

Binary Search Trees, Balanced BSTs

BST: Searching, Insertion
Hight Balance Property
Priority Queue: Introduction

Announcements/Reminders

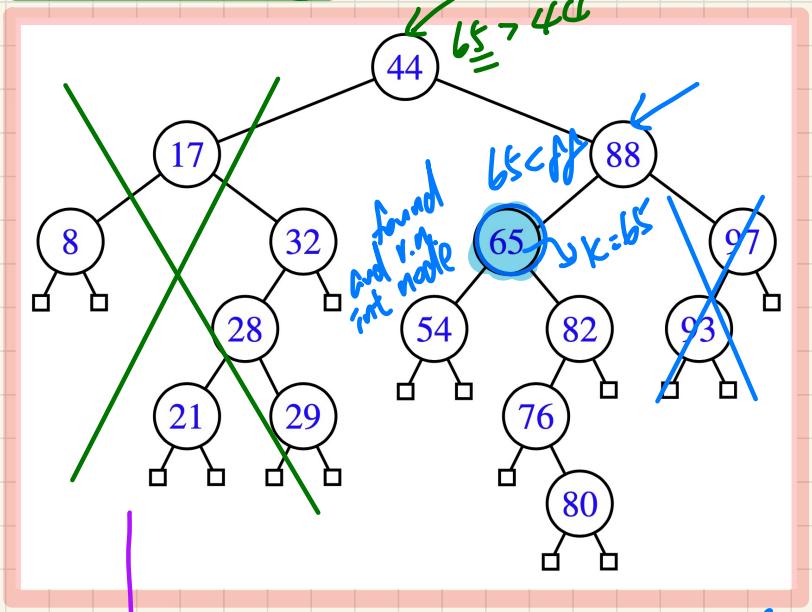
- Assignment 4 (on linked Trees) released
- Makeup Lecture (for ProgTest2) to be posted
- Bonus opportunity: Final Course Evaluation
- Office hours 3pm Tue/Wed/Thu this week
- Lecture notes template, Office Hours, TA Contact

* for a subsequent insertion, this ext. node can be set with a key value k s.t. $65 < k < 76$

BST Operation: Searching a Key

searching
① successful
↳ rtn. int. node with the matching key value
② unsuccessful
↳ rtn. ext. node with k: $65 < k < 76$

Search key 65

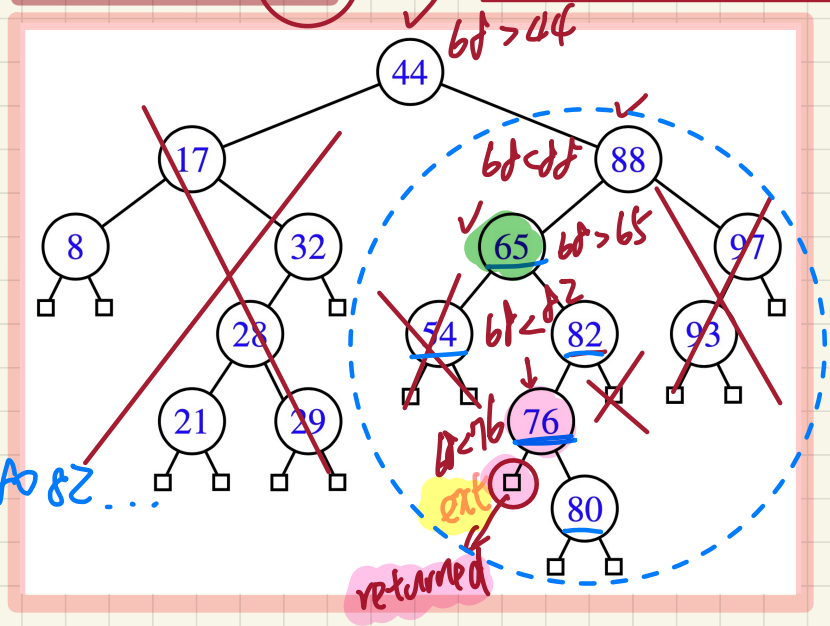


eliminate this subtree from search space (how large? 1/2 or O(1))

i.o.t. 54 65 76 A 82 ...

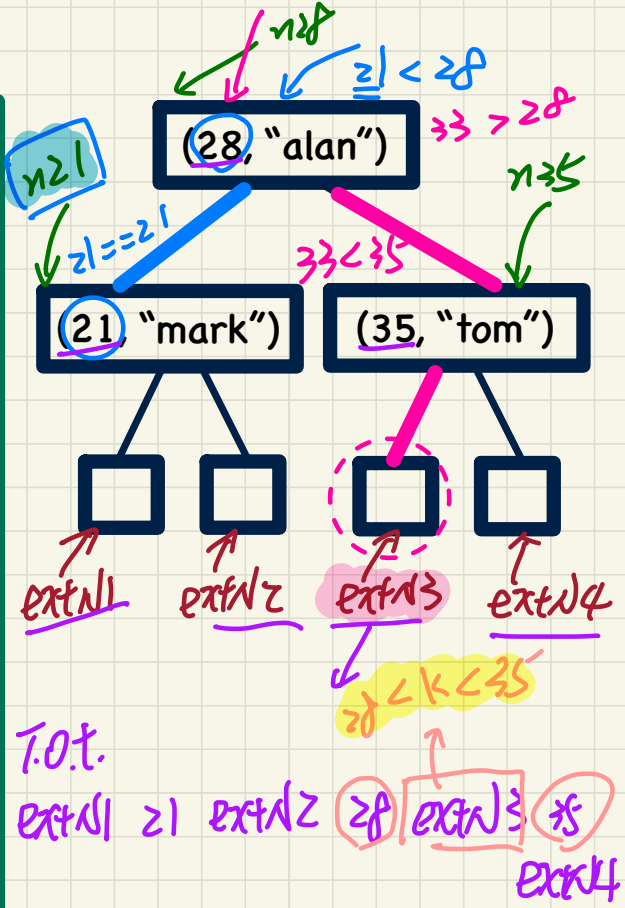
* ext.

Search key 68



Tracing: Searching through a BST

```
@Test
public void test_binary_search_trees_search() {
  BSTNode<String> n28 = new BSTNode<>(28, "alan");
  BSTNode<String> n21 = new BSTNode<>(21, "mark");
  BSTNode<String> n35 = new BSTNode<>(35, "tom");
  BSTNode<String> extN1 = new BSTNode<>();
  BSTNode<String> extN2 = new BSTNode<>();
  BSTNode<String> extN3 = new BSTNode<>();
  BSTNode<String> extN4 = new BSTNode<>();
  n28.setLeft(n21); n21.setParent(n28);
  n28.setRight(n35); n35.setParent(n28);
  n21.setLeft(extN1); extN1.setParent(n21);
  n21.setRight(extN2); extN2.setParent(n21);
  n35.setLeft(extN3); extN3.setParent(n35);
  n35.setRight(extN4); extN4.setParent(n35);

  BSTUtilities<String> u = new BSTUtilities<>();
  /* search existing keys */
  assertTrue(n28 == u.search(n28, 28));
  assertTrue(n21 == u.search(n28, 21));
  assertTrue(n35 == u.search(n28, 35));
  /* search non-existing keys */
  assertTrue(extN1 == u.search(n28, 17)); /* *17* < 21 */
  assertTrue(extN2 == u.search(n28, 23)); /* 21 < *23* < 28 */
  assertTrue(extN3 == u.search(n28, 33)); /* 28 < *33* < 35 */
  assertTrue(extN4 == u.search(n28, 38)); /* 35 < *38* */
}
```



* Recall: given N nodes: ___ ≤ h ___

Running Time: Search on a BST

```
public BSTNode<E> search(BSTNode<E> p, int k) {
  BSTNode<E> result = null;
  if(p.isExternal()) {
    result = p; /* unsuccessful search */
  }
  else if(p.getKey() == k) {
    result = p; /* successful search */
  }
  else if(k < p.getKey()) {
    result = search(p.getLeft(), k);
  }
  else if(k > p.getKey()) {
    result = search(p.getRight(), k);
  }
  return result;
}
```

root

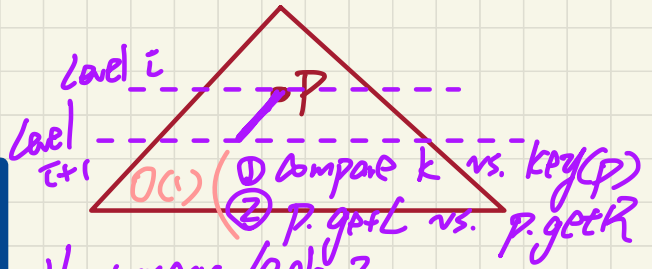
key to search

ext. node

ext. node

R1

R2



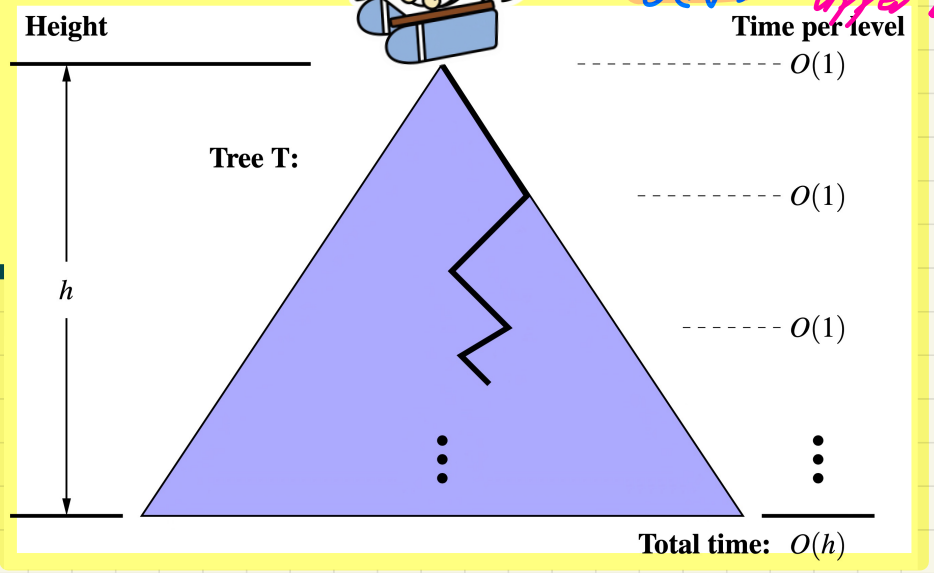
How many levels?

tight, attainable

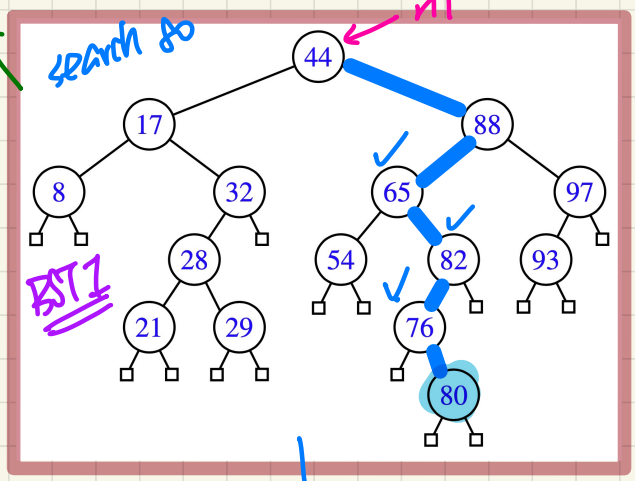
(A1) * O(h)

(A2) O(log N) → best case, lower bound

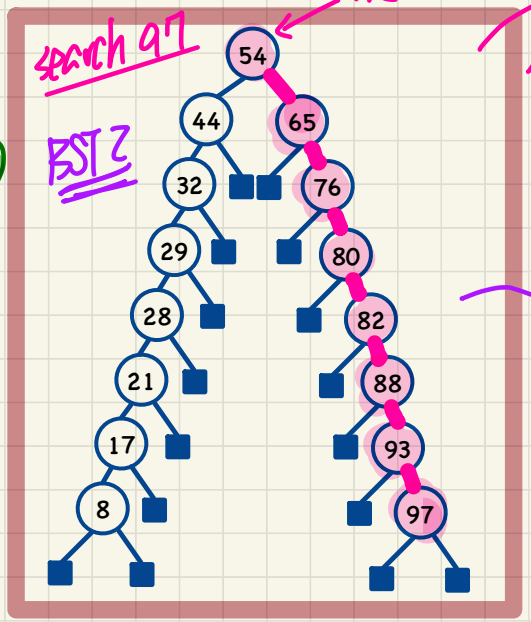
(A3) O(N) → worst case, upper bound



Binary Search: Non-Linear vs. Linear Structures



N=15
h=4
≈ O(log N)



unbalanced
N=15
h=7 ≈ N/2
O(N)

searching 97 on BST2 ≈ performing a linear search on a.

search 80 on BST1 ≈ performing a binary search on a

T.O.t.(n1) = T.O.t(n2) = a

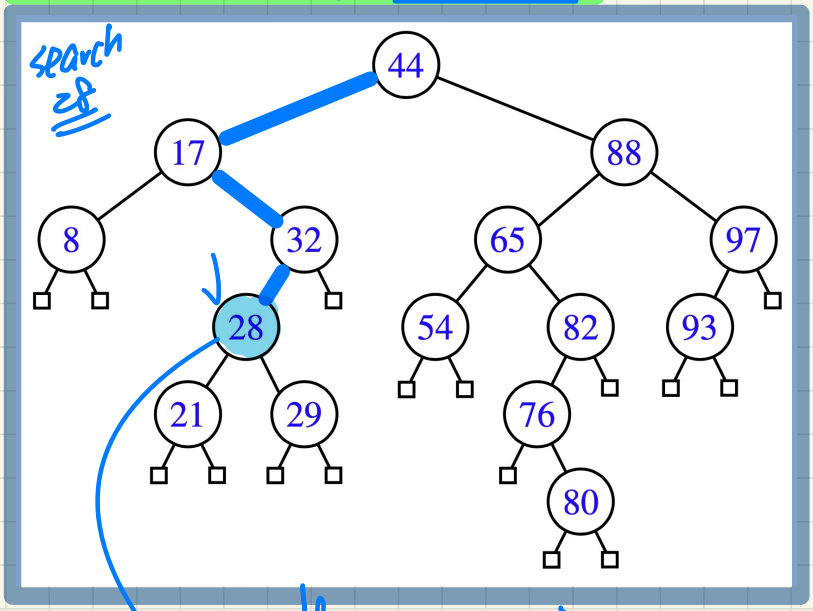
a

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
8	17	21	28	29	32	44	54	65	76	80	82	88	93	97

REVIEW !

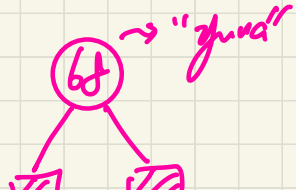
Visualizing BST Operation: Insertion

Insert Entry (28, "suyeon")

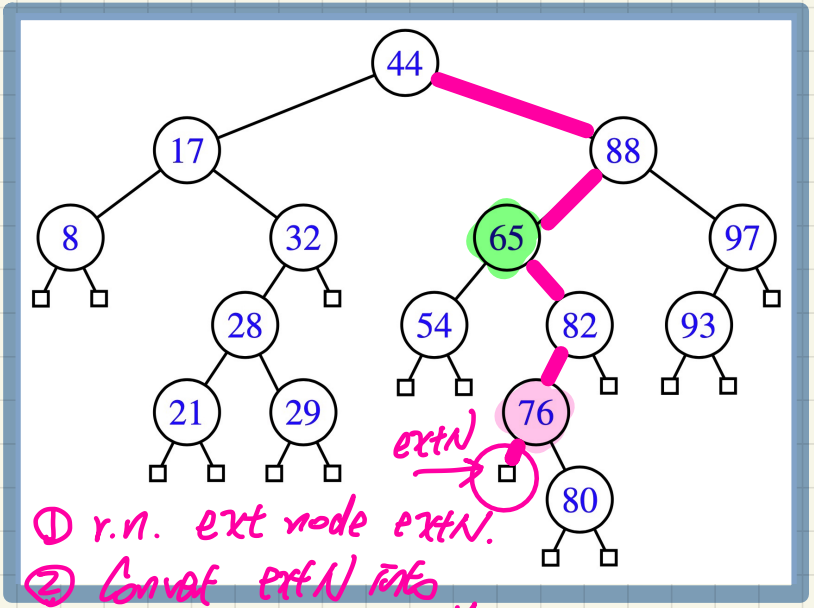


① r.n. int. node with key 28. node.setEle("suyeon");

②



Insert Entry (68, "yuna")



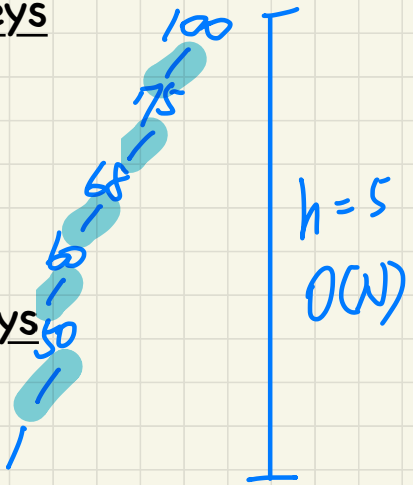
① r.n. ext node extN.

② Convert extN into

Worst-Case RT: BST with Linear Height

Example 1: Inserted Entries with Decreasing Keys

<100, 75, 68, 60, 50, 1>



Example 2: Inserted Entries with Increasing Keys

<1, 50, 60, 68, 75, 100>

entries (height? N vs. log N ?)

Example 3: Inserted Entries with In-Between Keys

<1, 100, 50, 75, 60, 68>

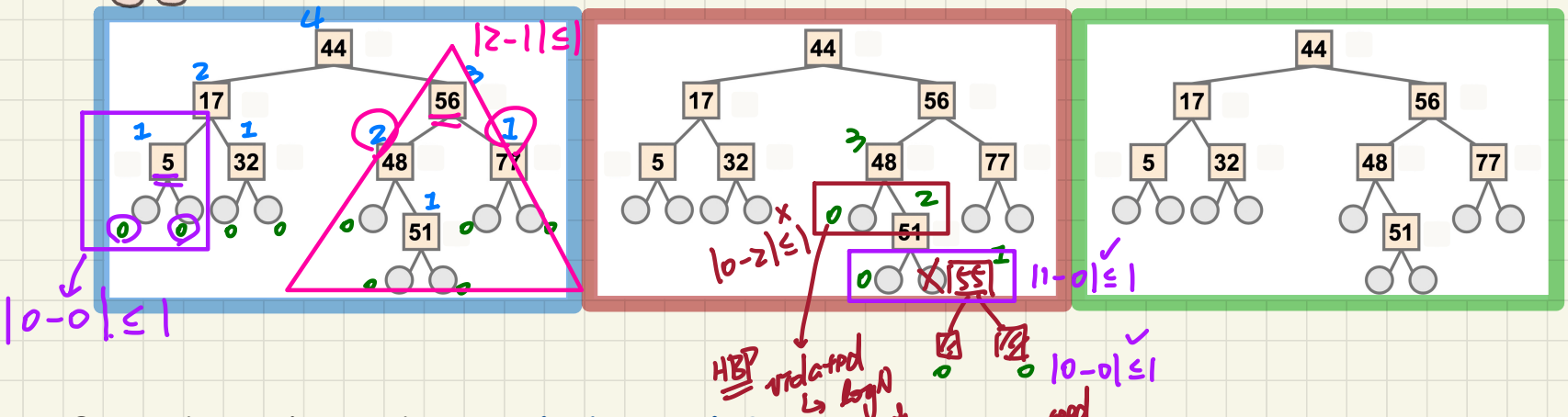
BST with O(N) height
⇒ search/insert/delete can be O(N).

Balanced BST: Definition

- internal node
- height
- height balance

Given a node p, the ***height*** of the subtree rooted at p is:

$$height(p) = \begin{cases} 0 & \text{if } p \text{ is external} \\ 1 + \mathbf{MAX}(\{ height(c) \mid parent(c) = p \}) & \text{if } p \text{ is internal} \end{cases}$$

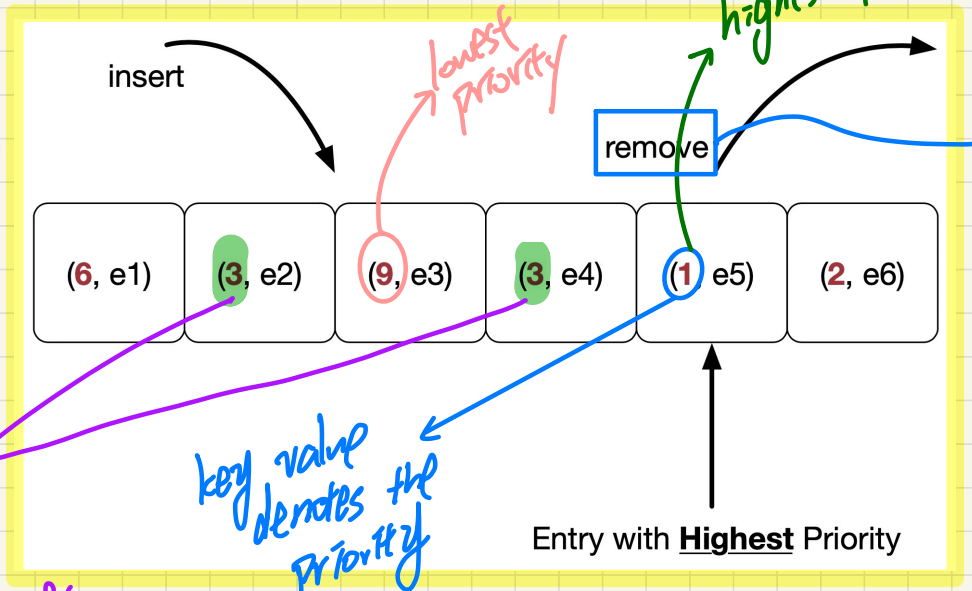


Q. Is the above tree a balanced BST?

Q. Still a balanced BST after inserting 55?

Q. Still a balanced BST after inserting 63?

What is a Priority Queue (PQ)



remove entry with the highest priority.

key value denotes the priority

Entries with the same priority (does not matter which entry is chosen).

Compare PQ with FIFO queue

1. Entries removed from PQ according to priorities.
2. Entries removed from FIFO a.t. chronological order

Lecture 21 - April 3

Priority Queues, Heap, Heap Sort

PQ: List Implementations
Heap: Structure, Relational Properties
Heap: Insertion, Deletion
Heap Sort

Announcements/Reminders

- Assignment 4 (on linked Trees) released
- Makeup Lecture (for ProgTest2) to be posted
- Bonus opportunity: Final Course Evaluation
- Office hours 3pm Thu this week
- Office hours, review session, ex. questions to be releasd
- Lecture notes template, Office Hours, TA Contact

List-Based Implementations of Priority Queue (PQ)

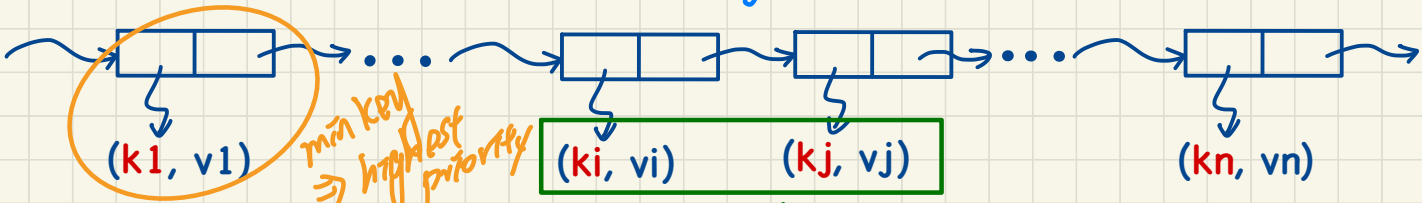
PQ Method	List Method	
	SORTED LIST	UNSORTED LIST
size, isEmpty	list.size O(1), list.isEmpty O(1)	
min	list.first O(1)	search min O(N)
insert	insert to "right" spot O(N) ≈ insertion sort	insert to front O(1)
removeMin	list.removeFirst O(1)	search min and remove O(N)

tracking space for temp

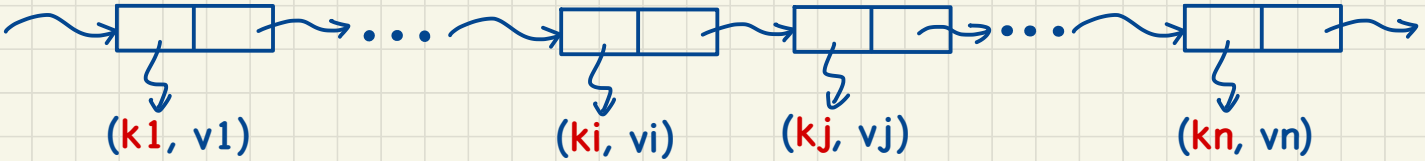
expansion of PQ. (2, infrequent

Approach 1: Sorted List

→ more suitable: (1) frequent retrieval/removal of top-priority entries.



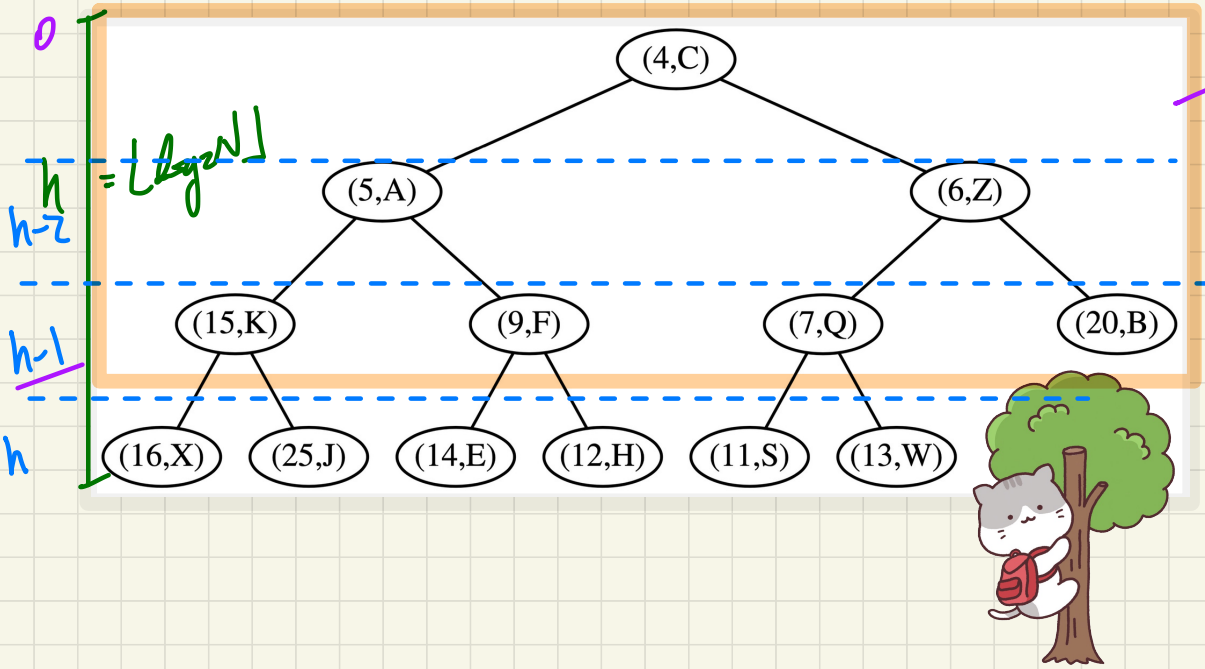
Approach 2: Unsorted List



Heaps: Structural Properties of Nodes

→ height: $\lfloor \log N \rfloor$.

Property: The tree is a complete Binary Tree



→ # nodes from levels 0 to h-1:

$2^h - 1$

Min # nodes:

$(2^h - 1) + 1$ 2^h

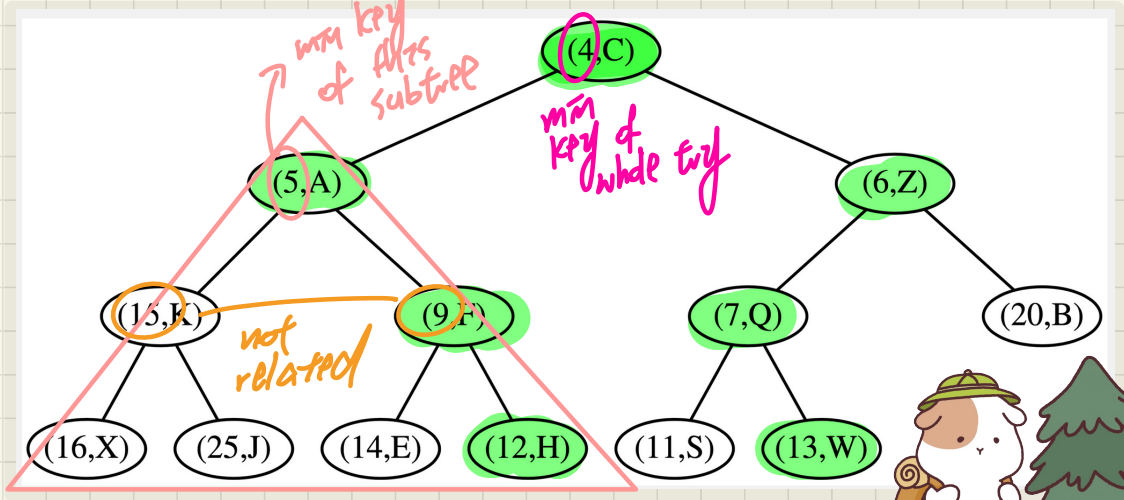
Max # nodes:

$(2^h - 1) + 2^h$

$2^{h+1} - 1$

Heaps: Relational Properties of Keys

Property: Each non-root node n is s.t. key(n) ≥ key(parent(n))



P1. Any leaf-to-root path has a sorted seq. of keys (non-ascending order).

P2. the min key exists in the root.

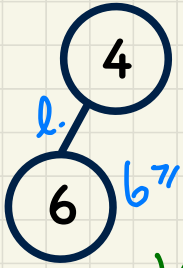
P3. key values between LST and RST are not related

Example Heaps

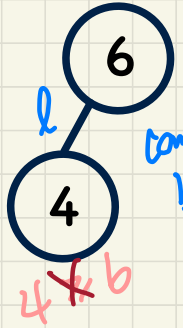
Example 1

smallest possible one-node heap.

Example 2

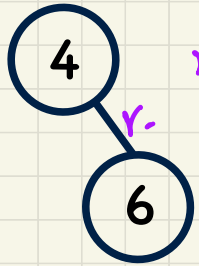


Example 3



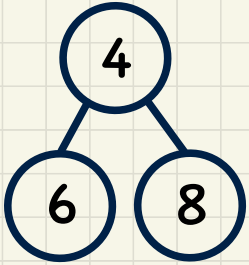
complete but violates H.O.P.

Example 4

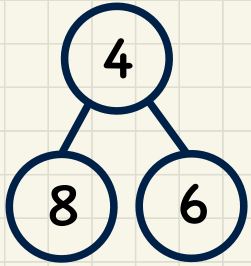


not complete but sat. H.O.P.

Example 5



Example 6



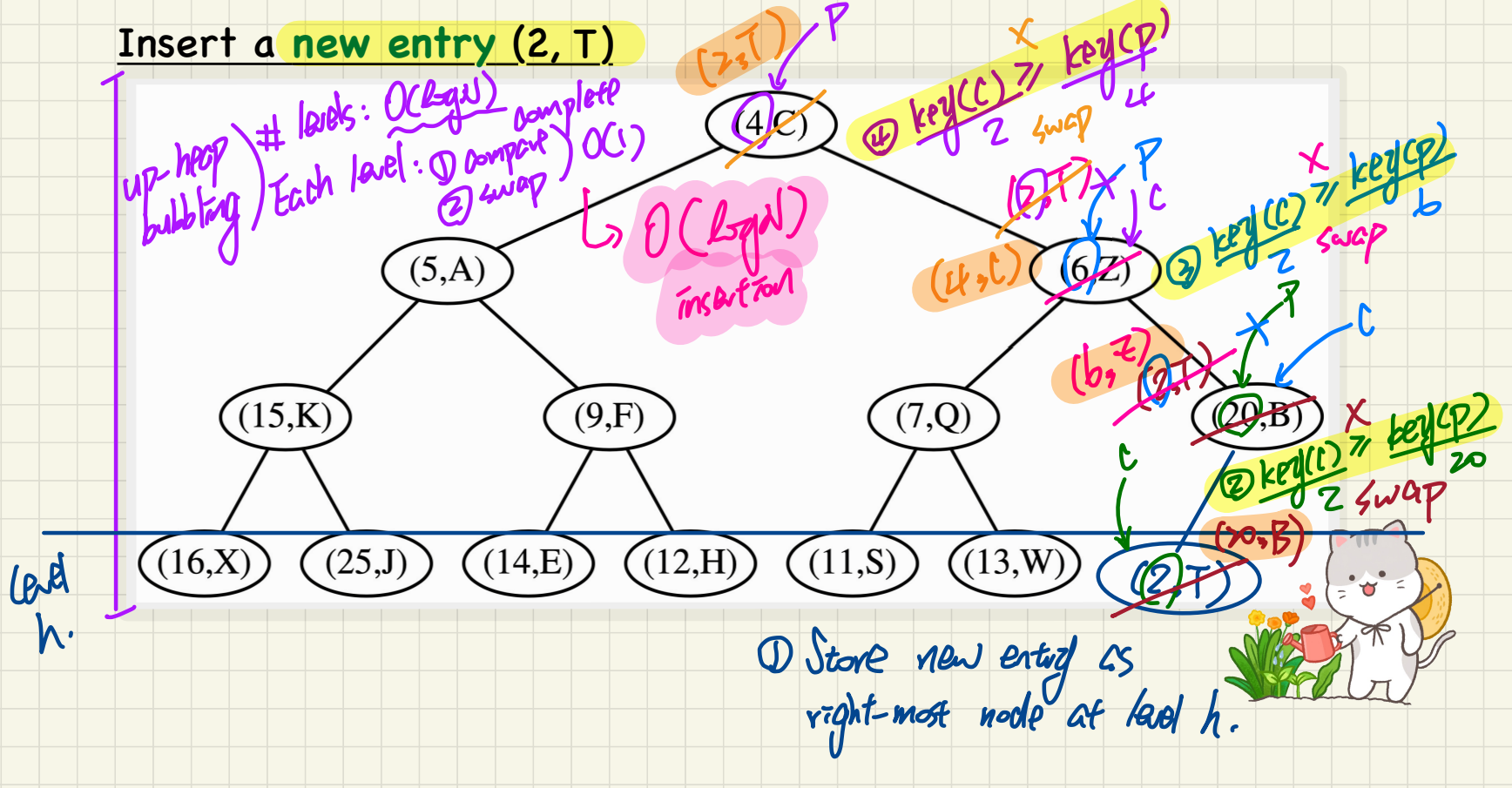
heaps!

full BT

⇒ complete BT.

Heap Operations: Insertion

Insert a new entry (2, T)

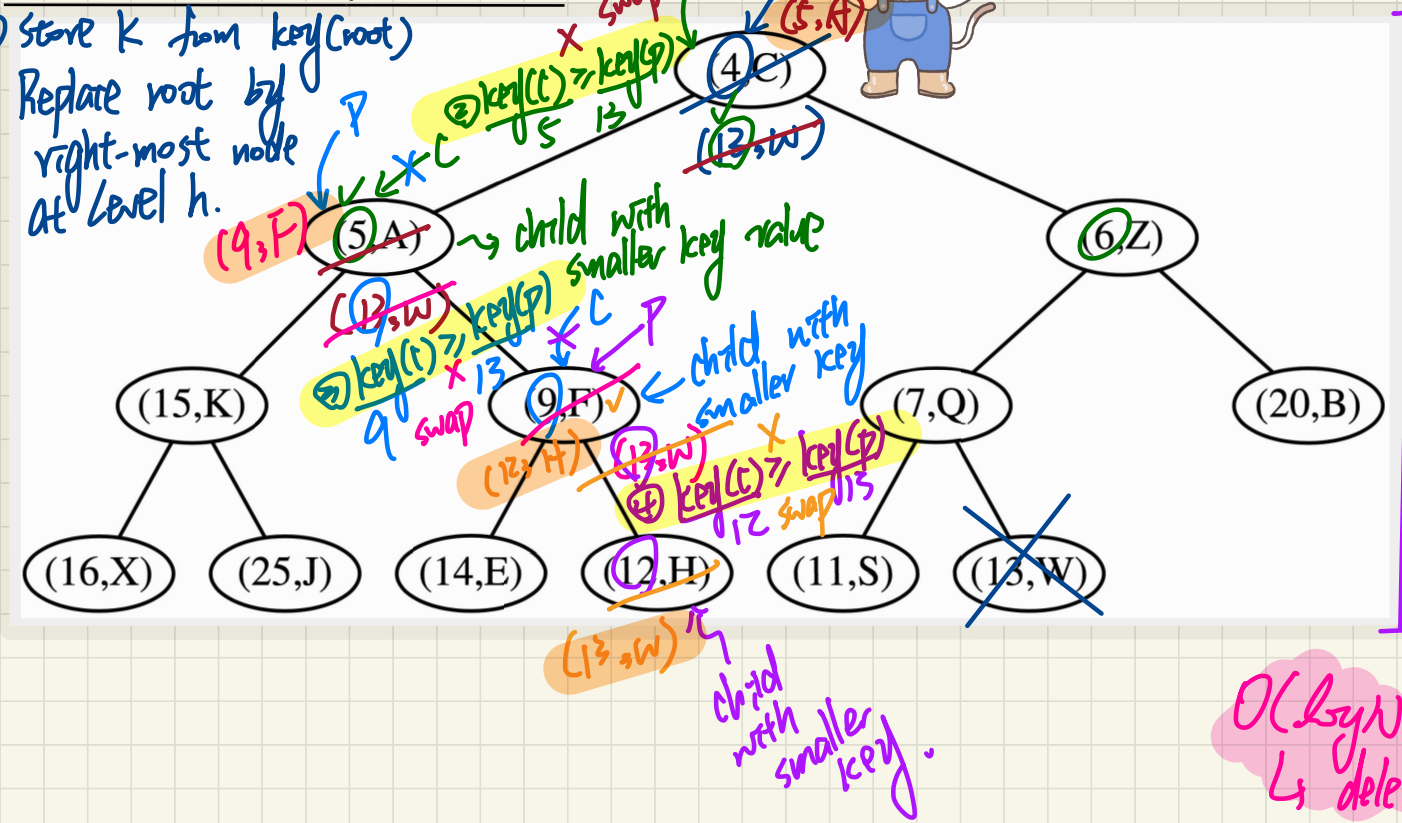


Heap Operations: Deletion

entry with min key (root).

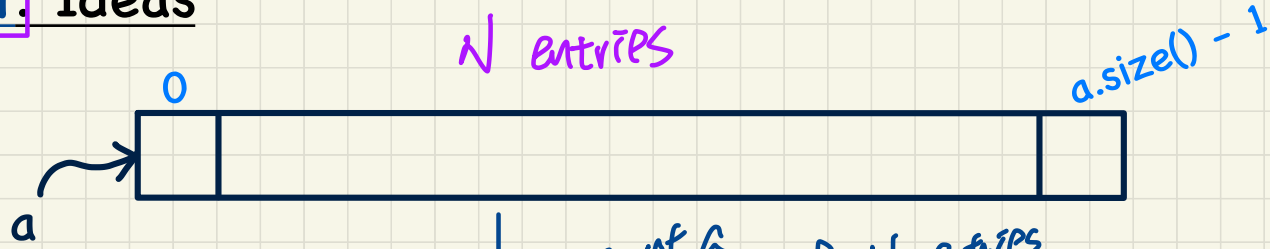
Delete the root/minimum

① store k from key(root)
Replace root by right-most node at level h.



$O(N \cdot \log N)$.

Heap Sort: Ideas



Construct a heap out of N entries

(A) Top-Down

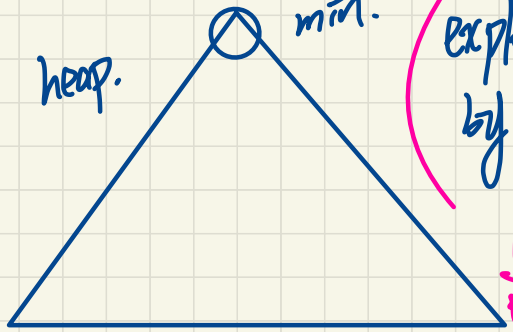
$O(N \cdot \log N)$.

(B) Bottom-Up

$O(N)$

$\approx$ selection sort

$O(N^2)$



(exploit the HOP (relational) by continuing to delete/extract min/root until tree is empty.

$O(N \cdot \log N)$ → heap sort.

N deletions, $\log N$ each deletion

Exam

~ 1~2 review sessions ~ Jeff, Sunda, Jackie

~ PDF guide

~ slides / iPad Notes

~ question booklet (answer booklets)

~ example code
↳ RSTNode.

↳ no calculator

~ assignments.

↳ little to none multiple choice questions

↳ definitions

↳ short answer (explanations, justification)

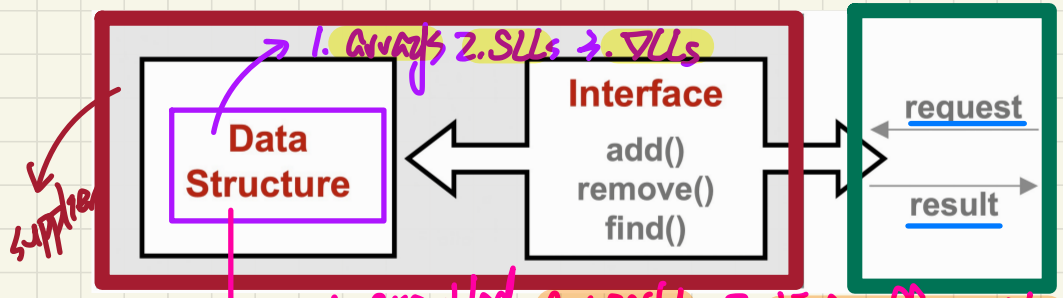
↳ coding / tracing

↳ proofs → e.g. asymptotic u.b.

Makeup Lecture (ProgTest1)

ADTs, Stacks

Abstract Data Types (ADTs)



client relies on the public interface of ADT()
1. input types
2. output types
3. what to be expected on IO related.

```
class Microwave {
  private boolean on;
  private boolean locked;
  void power() {on = true;}
  void lock() {locked = true;}
  void heat(Object stuff) {
    /* Assume: on && locked */
    /* stuff not explosive. */
  } }
```

```
class MicrowaveUser {
  public static void main(...) {
    Microwave m = new Microwave();
    Object obj = ???;
    m.power(); m.lock();]
    m.heat(obj);
  } }
```

	benefits	obligations
CLIENT	obtain a service	follow instructions
SUPPLIER	assume instructions followed	provide a service

Java API ≈ Abstract Data Types

NT is subject to ambiguities & contradictions.

E	**set**(int index, E element) Replaces the element at the specified position in this list with the specified element (optional operation).

set

```
E set(int index,
      E element)
```

Replaces the element at the specified position in this list with the specified element (optional operation).

Parameters:

index - index of the element to replace

element - element to be stored at the specified position

Returns:

the element previously at the specified position

Throws:

UnsupportedOperationException - if the set operation is not supported by this list

ClassCastException - if the class of the specified element prevents it from being added to this list

NullPointerException - if the specified element is null and this list does not permit null elements

IllegalArgumentException - if some property of the specified element prevents it from being added to this list

IndexOutOfBoundsException - if the index is out of range (index < 0 || index >= size())

Interface List<E>

Type Parameters:

E - the type of elements in this list

All Superinterfaces:

Collection<E>, Iterable<E>

All Known Implementing Classes:

AbstractList, AbstractSequentialList, ArrayList, AttributeList, CopyOnWriteArrayList, LinkedList, RoleList, RoleUnresolvedList, Stack, Vector

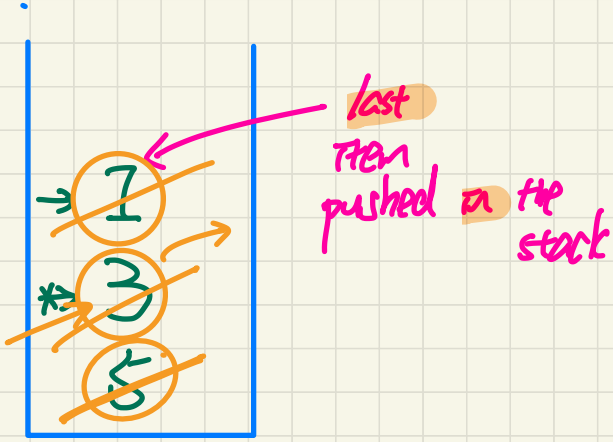
---

```
public interface List<E>
extends Collection<E>
```

An ordered collection (also known as a *sequence*). The user of this interface has precise control over where in the list each element is inserted. The user can access elements by their integer index (position in the list), and search for elements in the list.

Stack ADT: Illustration

	isEmpty	size	top
new stack	T	0	n.a.
push(5)	F	1	5
push(3)	F	2	3
push(1)	F	3	1
pop rtn. 1	F	2	3
pop rtn. 3	F	1	5
pop rtn. 5	T	0	n.a.



last pushed element

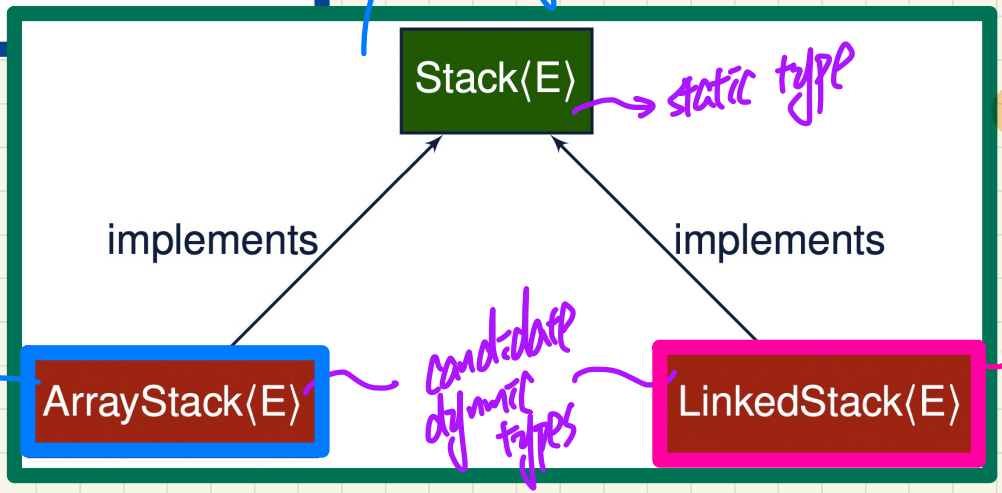
2nd last pushed element

(LIFO)

The order in which items are popped off the stack is the reverse of how these items were pushed.

Implementing the Stack ADT in Java: Architecture

```
public interface Stack<E> {
  public int size();
  public boolean isEmpty();
  public E top();
  public void push(E e);
  public E pop();
}
```



Implementing the Stack ADT using an Array

```
public class ArrayStack<E> implements Stack<E> {
  private final int MAX_CAPACITY = 1000;
  private E[] data;
  private int t; /* index of top */
  public ArrayStack() {
    data = (E[]) new Object[MAX_CAPACITY];
    t = -1;
  }

  public int size() { return (t + 1); }                    O(1)
  public boolean isEmpty() { return (t == -1); }           O(1)

  public E top() {                                         O(1)
    if (isEmpty()) { /* Precondition Violated */ }
    else { return data[t]; }
  }
  public void push(E e) {                                  O(1)
    if (size() == MAX_CAPACITY) { /* Precondition Violated */ }
    else { t ++; data[t] = e; }
  }
  public E pop() {                                         O(1)
    E result;
    if (isEmpty()) { /* Precondition Violated */ }
    else { result = data[t]; data[t] = null; t --; }
    return result;
  }
}
```

limitation: fixed size

ArrayStack<String>

↳ instantiates E for Stack:

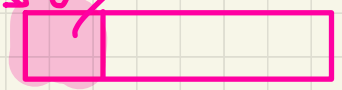
Stack<String>

(E[]) Object[____]

↳ what you have to write in Java.

efficient top. of stack

goal: treat this first item as the top.



Implementing the Stack ADT using a SLL

```
public class LinkedStack<E> implements Stack<E> {
  private SinglyLinkedList<E> list;
  ...
}
```

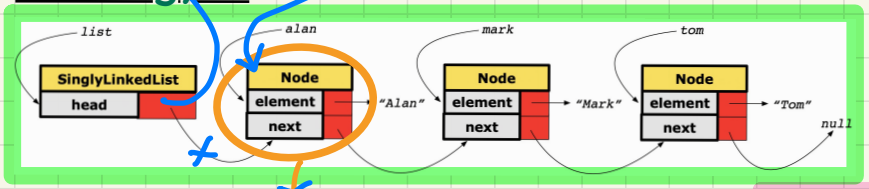
Stack Method	Singly-Linked List Method	
	Strategy 1	Strategy 2
size	list.size	
isEmpty	list.isEmpty	
top	list.first ✓	list.last
push	list.addFirst	list.addLast
pop	list.removeFirst	list.removeLast

Improved to O(1) if a DLL is used.

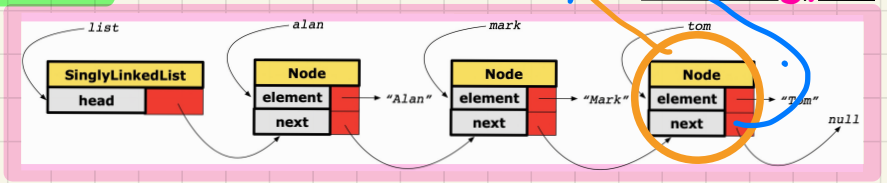
O(1)

O(1)

Strategy 1



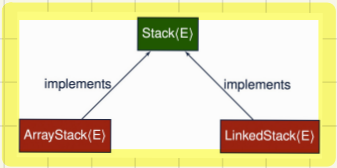
Strategy 2



Stack ADT: Testing Alternative Implementations

Stack<S> s = new Stack<>();

✗ interface can't be a DT.




```
public class ArrayStack<E> implements Stack<E> {
  private final int MAX_CAPACITY = 1000;
  private E[] data;
  private t; /* index of top */
  public ArrayStack() {
    data = (E[]) new Object[MAX_CAPACITY];
    t = -1;
  }

  public int size() { return (t + 1); }
  public boolean isEmpty() { return (t == -1); }

  public E top() {
    if (isEmpty()) { /* Precondition Violated */ }
    else { return data[t]; }
  }
  public void push(E e) {
    if (size() == MAX_CAPACITY) { /* Precondition Violated */ }
    else { t ++; data[t] = e; }
  }
  public E pop() {
    E result;
    if (isEmpty()) { /* Precondition Violated */ }
    else { result = data[t]; data[t] = null; t --; }
    return result;
  }
}
```

static type

```
@Test
public void testPolymorphicStacks() {
  Stack<String> s = new ArrayStack<>();
  s.push("Alan"); /* dynamic binding */
  s.push("Mark"); /* dynamic binding */
  s.push("Tom"); /* dynamic binding */
  assertTrue(s.size() == 3 && !s.isEmpty());
  assertEquals("Tom", s.top());

  s = new LinkedStack<>();
  s.push("Alan"); /* dynamic binding */
  s.push("Mark"); /* dynamic binding */
  s.push("Tom"); /* dynamic binding */
  assertTrue(s.size() == 3 && !s.isEmpty());
  assertEquals("Tom", s.top());
}
```

dynamic type

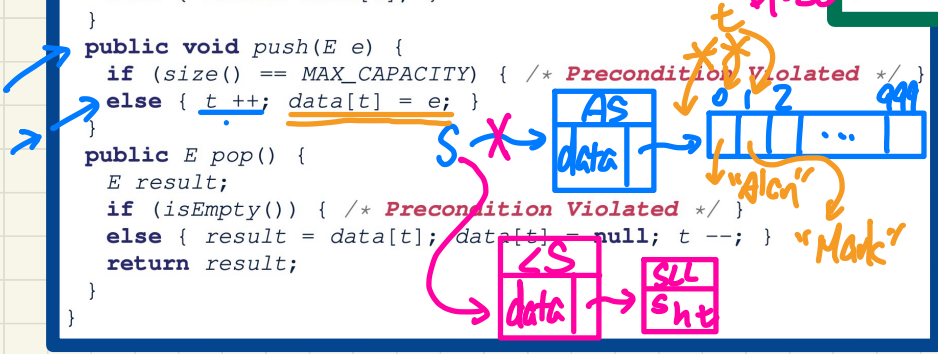
DT: AS.

version in AS.

DT changes

DT: LS

version in LS. class of Stack?



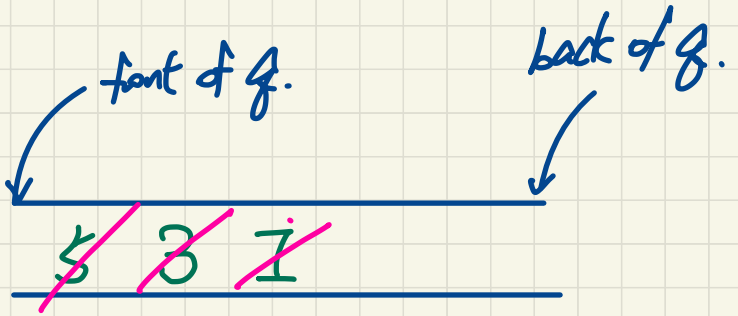
Is the DT of S a descendant of:		*	**
S	instanceof Stack	T	T
S	instanceof ArrayStack	T	F
S	instanceof LinkedStack	F	T

Makeup Lecture (WrittenTest)

Queues, Circular Arrays, Deque

Queue ADT: Illustration

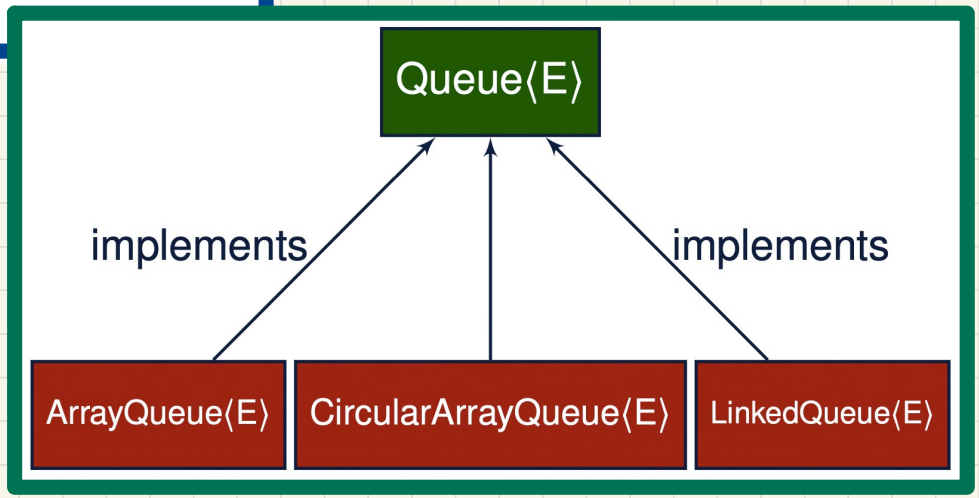
	isEmpty	size	first
new queue	T	0	n.a.
enqueue(5)	F	1	5
enqueue(3)	F	2	5
enqueue(1)	F	3	5
dequeue rtn. 5	F	2	3
dequeue rtn. 3	F	1	1
dequeue rtn. 1	T	0	n.a.



→ First-In First-Out (FIFO)

Implementing the Queue ADT in Java: Architecture

```
public interface Queue<E> {
   public int size();
   public boolean isEmpty();
   public E first();
   public void enqueue(E e);
   public E dequeue();
}
```



Implementing the Queue ADT using an Array

```
public class ArrayQueue<E> implements Queue<E> {
  private final int MAX_CAPACITY = 1000;
  private E[] data;
  private int r; /* rear index */
  public ArrayQueue() {
    data = (E[]) new Object[MAX_CAPACITY];
    r = -1;
  }
  public int size() { return (r + 1); }
  public boolean isEmpty() { return (r == -1); }
  public E first() {
    if (isEmpty()) { /* Precondition Violated */ }
    else { return data[0]; }
  }
  public void enqueue(E e) {
    if (size() == MAX_CAPACITY) { /* Precondition Violated */ }
    else { r ++; data[r] = e; }
  }
  public E dequeue() {
    if (isEmpty()) { /* Precondition Violated */ }
    else {
      E result = data[0];
      for (int i = 0; i < r; i ++) { data[i] = data[i + 1]; }
      data[r] = null; r --;
      return result;
    }
  }
}
```

O(1)

O(1)

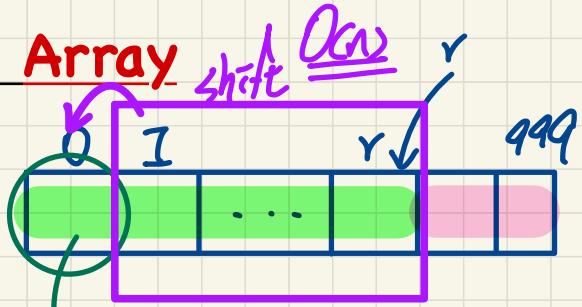
O(1)

front index

O(1)

O(n)

shifting "2nd item" and onwards to the left by one position



front of queue

limitation: no resizing.

to improve this, we need to be flexible about where the front index is ⇒ Circular array.

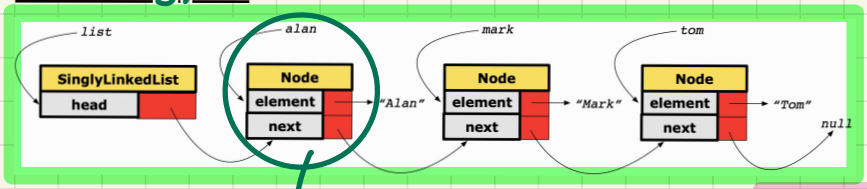
Implementing the Queue ADT using a SLL

```
public class LinkedQueue<E> implements Queue<E> {
  private SinglyLinkedList<E> list;
  ...
}
```

O(n)
① use SI instead
② use DLL instead

Queue Method	Singly-Linked List Method	
	Strategy 1	Strategy 2
size	list.size	
isEmpty	list.isEmpty	
first	list.first	list.last
enqueue	list.addLast	list.addFirst
dequeue	list.removeFirst	list.removeLast

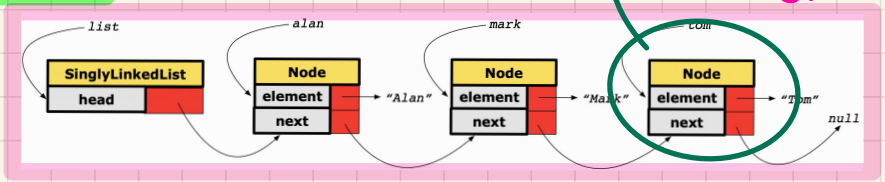
Strategy 1



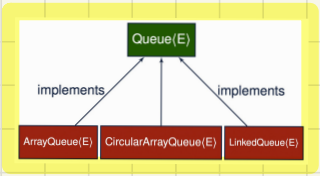
first of queue

first of queue.

Strategy 2



Stack ADT: Testing Alternative Implementations




```
public class ArrayQueue<E> implements Queue<E> {
  private final int MAX_CAPACITY = 1000;
  private E[] data;
  private int r = -1; /* rear index */
  public ArrayQueue() {
    data = (E[]) new Object[MAX_CAPACITY];
    r = -1;
  }
  public int size() { return (r + 1); }
  public boolean isEmpty() { return (r == -1); }
  public E first() {
    if (isEmpty()) { /* Precondition Violated */ }
    else { return data[0]; }
  }
  public void enqueue(E e) {
    if (size() == MAX_CAPACITY) { /* Precondition Violated */ }
    else { r ++; data[r] = e; }
  }
  public E dequeue() {
    if (isEmpty()) { /* Precondition Violated */ }
    else {
      E result = data[0];
      for (int i = 0; i < r; i ++) { data[i] = data[i + 1]; }
      data[r] = null; r --;
      return result;
    }
  }
}
```

dynamic binding.

```
@Test
public void testPolymorphicQueues() {
  Queue<String> q = new ArrayQueue<>();
  q.enqueue("Alan"); /* dynamic binding */
  q.enqueue("Mark"); /* dynamic binding */
  q.enqueue("Tom"); /* dynamic binding */
  assertTrue(q.size() == 3 && !q.isEmpty());
  assertEquals("Alan", q.first());

  q = new LinkedQueue<>();
  q.enqueue("Alan"); /* dynamic binding */
  q.enqueue("Mark"); /* dynamic binding */
  q.enqueue("Tom"); /* dynamic binding */
  assertTrue(q.size() == 3 && !q.isEmpty());
  assertEquals("Alan", q.first());
}
```

Polymorphism

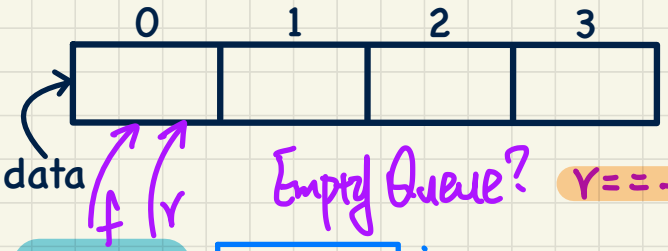
size of Queue vs. size of array | % modulo

Implementing the Queue ADT using a Circular Array

1. fix-sized (no resizing)
2. flexible for performing "dequeue"

Assume: A circular array of length 4.

Phase 0: Empty Queue q



f=0;
r=0;

Empty Queue? r==f

Phase 1: enqueue 3 elements

- data[r] = item;
- r++;

Queue Full? (r+1)%N == f
when r points to is the only empty slot.

(r+1)%4 = 3

(r+1)%4 = 4 = 0

[f, r-1] = (r-1)-f+1 = r-f

empty slots before f | # slots before this index

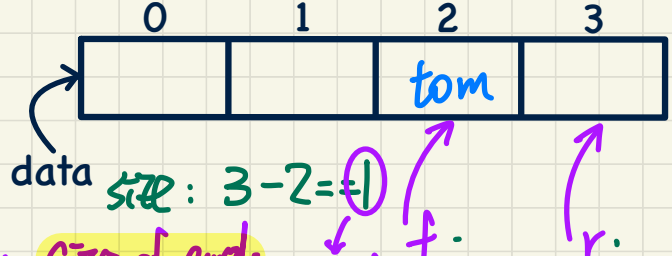
size: r>=f → r-f

3 = 1+(4-2)

size: r<f → r+(N-f)

Phase 2: dequeue 2 times

data[f]=null; f++;



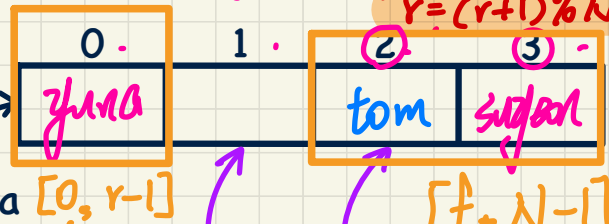
size: 3-2=1

size of array: 4 (fixed) → size of 4.

(3+1)%4 = 0

Phase 3: enqueue 2 elements

data[r] = item; ~~r++;~~ r=(r+1)%N



[0, r-1] → (r-1)-0+1 = r

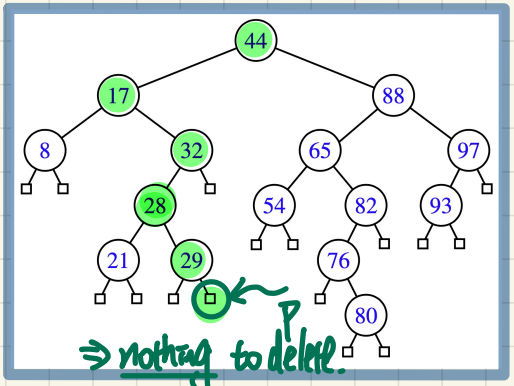
[f, N-1] → (N-1)-f+1 = N-f

Makeup Lecture (ProgTest2)

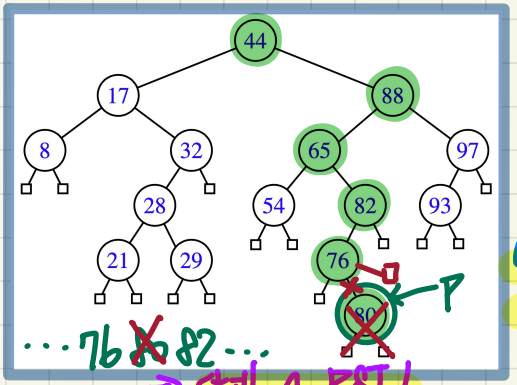
BST Deletions, Constructing Heap, Array-Based Implementations of BTs

Visualizing BST Operation: Deletion

Case 1: Delete Entry with Key 31

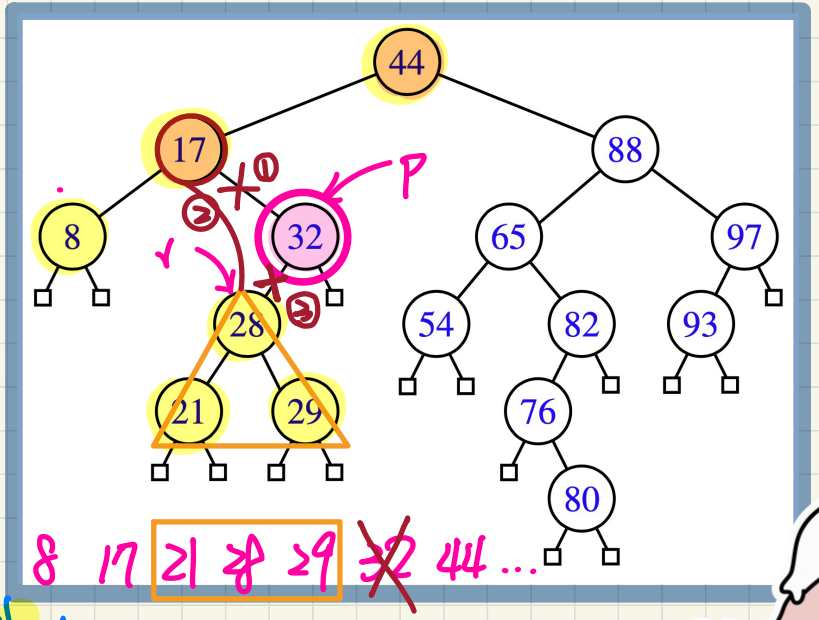


Case 2: Delete Entry with Key 80



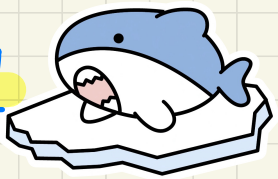
Orphan subtree: a subtree rooted at an internal child node

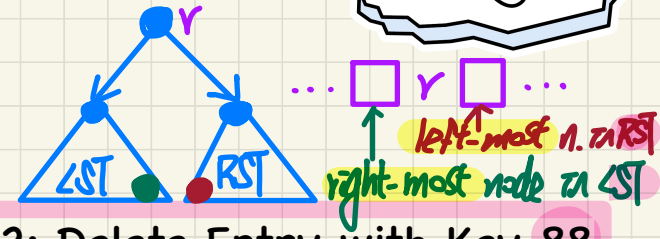
Case 3: Delete Entry with Key 32



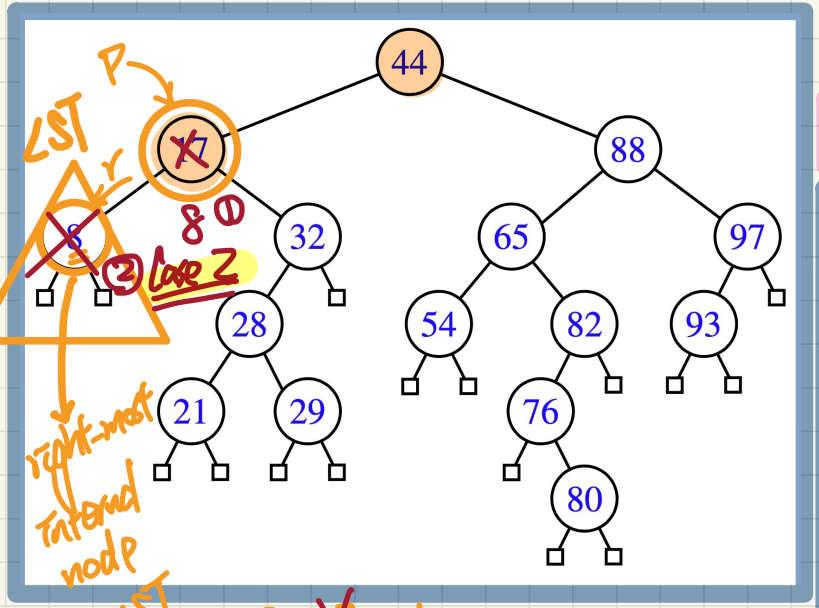
Visualizing BST Operation: Deletion

In-Order Traversal

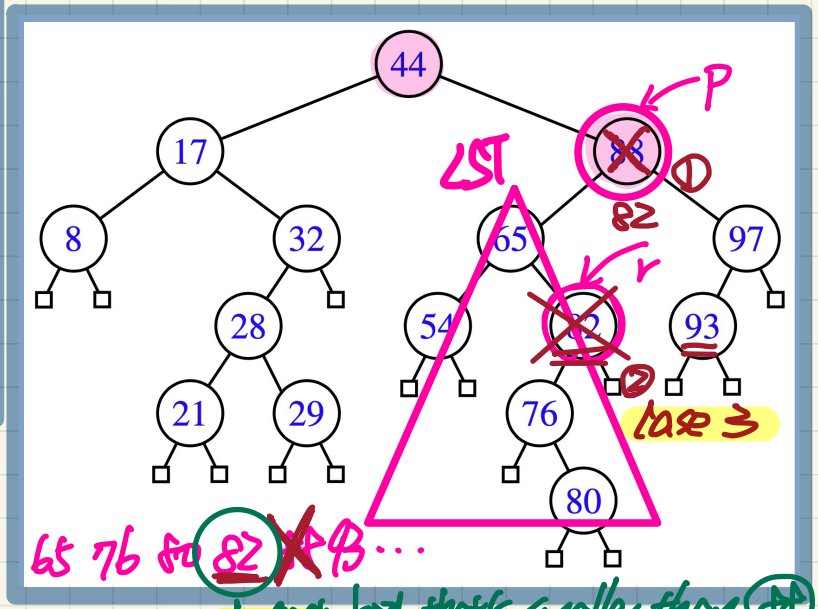




Case 4.1: Delete Entry with Key 17



Case 4.2: Delete Entry with Key 88



Top-Down Heap Construction

Problem: Build a heap out of ***N*** entires, supplied one at a time.

- Initialize an ***empty heap h***.
- As each new entry **e** = (*k*, *v*) is supplied, **insert** **e** into ***h***.

Exercise: Build a heap out of the following 15 keys:

<16, 15, 4, 12, 6, 7, 23, 20, 25, 9, 11, 17, 5, 8, 14>

first inserted to level 2

Assumption: Key values supplied **one at a time**.

first inserted to level 1

$* \le 1 + \log_2 n \cdot (2^1 + 2^2 + \dots + 2^h)$

$= 1 + \log_2 n \,(n-1)$

$O(n \cdot \log n)$

RI: # nodes at level i # up-heap bubbling steps

$1 + 2^1 \cdot 1 \le \log n$

$+ 2^2 \cdot 2 \le \log n$

$+ \dots$

$+ 2^h \cdot h \log n$

$2^0 = 1$ Level 0

$2^1 = 2$ Level 1

$2^2 = 4$ Level 2

2^h Level h

Exercise: Complete inserting the remaining keys to the heap.

Bottom-Up Heap Construction

Problem: Build a heap out of ***N*** entires, supplied all at once.

- **Assume**: The resulting heap will be ***completely filled*** at **all** levels.
  $\Rightarrow N = 2^{h+1} - 1$ for some ***height*** $h \geq 1$ $\quad [\, h = (\log(N + 1)) - 1 \,]$
- Perform the following steps called *Bottom-Up Heap Construction*:

  Step 1: Treat the first $\frac{N+1}{2^1}$ list entries as heap roots.
  $\therefore \frac{N+1}{2^1}$ heaps with height 0 and size $2^1 - 1$ constructed.

  Step 2: Treat the next $\frac{N+1}{2^2}$ list entries as heap roots.
  - ◇ Each ***root*** sets two heaps from **Step 1** as its ***LST*** and ***RST***.
  - ◇ Perform ***down-heap bubbling*** to restore **HOP** if necessary.

  $\therefore \frac{N+1}{2^2}$ heaps, each with height 1 and size $2^2 - 1$ constructed.

  ...

  Step $h+1$: Treat next $\frac{N+1}{2^{h+1}} = \frac{(2^{h+1}-1)+1}{2^{h+1}} = 1$ list entry as heap root.
  - ◇ Each ***root*** sets two heaps from **Step h** as its ***LST*** and ***RST***.
  - ◇ Perform ***down-heap bubbling*** to restore **HOP** if necessary.

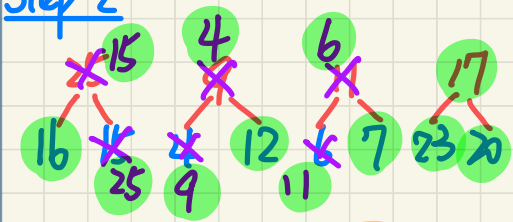
  $\therefore \frac{N+1}{2^{h+1}} = 1$ heap, each with height h and size $2^{h+1} - 1$ constructed.

Exercise: Build a heap out of the following 15 keys:
<16, 15, 4, 12, 6, 7, 23, 20, 25, 9, 11, 17, 5, 8, 14>

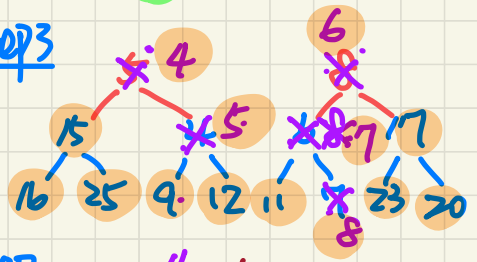
Assumption: Key values supplied **all at once**.

50% Step 1
8 heaps, size 1, height 0
16 15 4 12 6 7 23 20

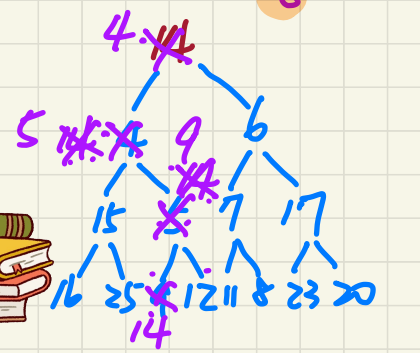
Step 2
25%



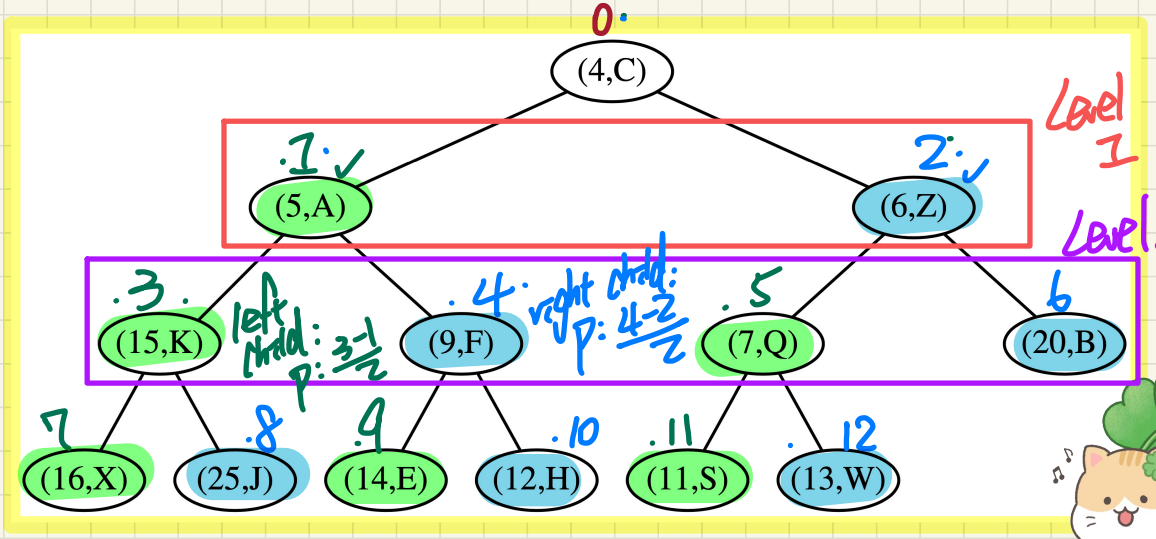
Step 3
12.5%



Step

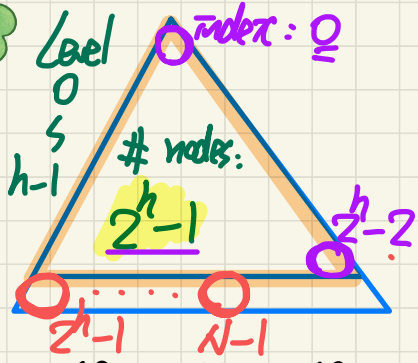


Array-Based Representation of a Complete BT



Exercise

What if the BT is not complete? (bad for space util.)



$$index(x) = \begin{cases} 0 & \text{if } x \text{ is the root} \\ 2 \cdot index(\, parent(x)\,) + 1 & \text{if } x \text{ is a left child} \\ 2 \cdot index(\, parent(x)\,) + 2 & \text{if } x \text{ is a right child} \end{cases}$$

0	1	2	3	4	5	6	7	8	9	10	11	12
(4,C)	(5,A)	(6,Z)	(15,K)	(9,F)	(7,Q)	(20,B)	(16,X)	(25,J)	(14,E)	(12,H)	(11,S)	(13,W)

I hope you enjoyed learning with me 🤗

All the best to you !